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BUNDLE SPACES FOR APPROXIMATE FIBRATIONS

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1. Introduction

Approximate fibration is a proper map having an approximate homotopy lifting property for all spaces, which is introduced by Coram and Duvall [1]. It is a generalization of Hurewicz fibration but it has very useful, analogous properties to the fibration such as the existence of homotopy exact sequence. Owing to this advantage, the problem that under what condition a proper map $q : M \to B$ is an approximate fibration has been an interesting issue [2,3,4,5,7,11].

In this paper, we are going to suggest closed manifolds N with bundle structures which force maps $q : M \to B$ to be approximate fibrations, when M is an $(n+2)$-manifold and each $q^{-1}b$ has the homotopy type of N.

A proper map $q : M \to B$ between locally compact ANR's is called an *approximate fibration* if it has the following approximate homotopy lifting property: given an open cover ε of B, an arbitrary space X, and two maps $h : X \to M$ and $F : X \times I \to B$ such that $q \circ h = F_0$, there exists a map $H : X \times I \to M$ such that $H_0 = h$ and $q \circ H$ is ε-close to F.

If $q : M \to B$ is an approximate fibration to a path connected space B, then point inverses are absolute neighborhood retracts and pairwise homotopy equivalent. A branch of the research on approximate fibrations is to find out conditions for the fibers of p in the collection mentioned above. We assume all spaces are locally compact, metrizable ANR's, and all manifolds are finite dimensional, orientable, connected and boundaryless. A manifold M is said to be *closed* if M is compact.

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A closed n-manifold N is called a *codimension k fibrator* if, whenever $q : M \to B$ is a proper map from an arbitrary $(n+k)$-manifold M to a finite dimensional space B such that each point preimage $q^{-1}(b)$ is homotopic equivalent to N, $q : M \to B$ is an approximate fibration.

All closed aspherical n-manifolds N for which $\pi_1(N)$ is finitely generated are codimension 1-fibrators. Actually, the information about codimension 1-fibrator is almost known. All surfaces except those of Euler characteristic zero and their product are codimension 2-fibrators [4,11]. Every $(k-1)$-connected closed manifold N is a codimension k-fibrator but n-sphere S^n is not a codimension $(n+1)$-fibrator.

The *degree* of a map $R : N \to N$, where N is a closed manifold, is the nonnegative integer d such that the induced endomorphism of $H_n(N;\mathbb{Z}) \cong \mathbb{Z}$ amounts to multiplication by d, up to sign. Note that a degree one map $R : N \to N$ induces homology isomorphisms $R_* : H_i(N) \to H_i(N)$ for all integer $i \geq 0$ [16] and the epimorphism $R_\# : \pi_1(N) \to \pi_1(N)$ [10].

The *continuity set* C of $q : M \to B$ consists of those points $c \in B$ such that, under any retraction $R : q^{-1}U \to q^{-1}c$ defined over a neighborhood $U \subset B$ of c, c has another neighborhood $V_c \subset U$ such that $R|_{q^{-1}b} : q^{-1}b \to q^{-1}c$ is a degree one map for all $b \in V_c$. Coram and Duvall [3] showed that the continuity set of q is a dense, open subset of B

2. Hopfian manifolds with bundle structure

A closed manifold N is called *Hopfian* if every degree one map $N \to N$ which induces a π_1-automorphism is a homotopy equivalence. Davermann [5] proved that if N is a closed Hopfian manifold then an N-like decomposition map is an approximate fibration on the continuity set. So whether a closed manifold N is Hopfian is a part of a significant problem in efficiently identifying codimension k-fibrators. We mainly investigate manifolds to be Hopfian manifolds in this section.

A fiber bundle (N, E, F, p) consists of the bundle space N, a base space E, the fiber F, and a bundle projection $p : N \to E$ such that there exists an open covering $\mathcal{U}$ of E and, for each $U \in \mathcal{U}$, a homeomorphism $\psi_U : U \times F \to p^{-1}(U)$ such that the composite

$$U \times F \xrightarrow{\psi_U} p^{-1}(U) \xrightarrow{p} U$$

is the projection to the first factor. We denote the bundle space N by $E\tilde{\times}F$ in the sense of comparison to product space $E \times F$.

LEMMA2.1 [19]. *Let $R : M_1 \to M_2$ be a map of closed n-manifolds which induces an isomorphism in the fundamental groups. Suppose that $\pi_i(M_1)$ and $\pi_i(M_2)$ are trivial for $1 < i < n-1$. R is a homotopy equivalence if and only if the degree of R is ± 1. In particular, an aspherical closed manifold is a Hopfian manifold.*

THEOREM 2.2. *If F_1 and F_2 are aspherical closed manifolds then the bundle $F_1\tilde{\times}F_2$ is a Hopfian manifold.*

Proof. Let $p : F_1\tilde{\times}F_2 \to F_1$ be the bundle projection. Since the base space F_1 is a compact manifold, p is a fibration and so there exists the homotopy exact sequence between three objects;

$$\cdots \to \pi_n(F_2) \to \pi_n(F_1\tilde{\times}F_2) \to \pi_n(F_1) \to \pi_{n-1}(F_2) \to \cdots$$

From the fact that F_1 and F_2 are aspherical, the above homotopy sequence provides the information that $F_1\tilde{\times}F_2$ is also aspherical. By Lemma 2.1, $F_1\tilde{\times}F_2$ is a Hopfian manifold.

A group H is called *hopfian* if every epimorphism $\Psi : H \to H$ is necessarily an isomorphism. Sometimes, the Hopfian property of fundamental group of a closed manifold makes N a Hopfian manifold. For low dimensional manifold, Hausmann proved the following useful result;

LEMMA 2.3 [8]. *A closed, orientable n-manifold N is a Hopfian manifold provided $n \leq 4$ and $\pi_1(N)$ is Hopfian.*

THEOREM 2.4. *Let F_1 and F_2 be closed surfaces with nonzero Euler characteristics. If F_1 or F_2 is aspherical, then the bundle N over F_1 with a fiber F_2 is a Hopfian manifold.*

Proof. If both F_1 and F_2 are aspherical, by Theorem 2.2, we are done. Otherwise, one of them is not aspherical, say F_1. By the virtue of the classification theorem for compact surface, it is homeomorphic to a sphere S^2. A bundle space N over 2-manifold having 2-manifold as a fiber is a 4-manifold and thus it suffices to show that N has a Hopfian fundamental group by the view of Lemma 2.3.

Consider the following homotopy exact sequence

$$0 \cong \pi_2(F_2) \to \pi_2(N) \xrightarrow{p_\#} \pi_2(F_1) \xrightarrow{g} \pi_1(F_2) \xrightarrow{i_\#} \pi_1(N) \to \pi_1(F_1) \to 1$$

Since $p_\#$ is one to one and $\pi_2(F_1) \cong \pi_2(S^2) \cong \mathbb{Z}$, $\pi_2(N)$ is a free group. From the fact that $\pi_2(F_1) \cong \pi_2(N) \oplus$ Im g and the subgroup Im g of a free group $\pi_1(F_2)$ is free, Im g must be a trivial group. It ensures that $i_\# : \pi_1(F_2) \to \pi_1(N)$ is an isomorphism. On the other hand, the fundamental group of an aspherical manifold is torsion free and thus $\pi_1(F_2)$ is a finitely generated free group, which implies that it is hopfian. Therefore $\pi_1(N)$ is a Hopfian group.

In [12], Im, Kang and Woo showed that a product $S^n \times F$ of an n-sphere $S^n (n > 1)$ and a closed aspherical manifold F is a Hopfian manifold. We are going to extend the result to the bundle space $S^n \tilde{\times} F$.

By a *section* of a bundle projection $p : N \to E$ we mean a continuous map $f : E \to N$ such that $p \circ f(x) = x$ for each $x \in E$. Note that every product space $N = S \times F$ is a bundle having sections, which are just the graphs of maps $E \to F$.

THEOREM 2.5. *Let N be a closed bundle $S^n \tilde{\times} F$ over an n-sphere S^n having an aspherical closed $k(n > k \geq 2)$-manifold F as a fiber. If there is a section $f : S^n \to N$ of the bundle projection $p : N \to S^n$ then N is a Hopfian manifold.*

Proof. Let $R : N \to N$ be a degree one map inducing a π_1-isomorphism and let R_* and $R_\#$ be the induced endomorphisms by R on homology groups $H_i(N)$ and homotopy groups $\pi_i(N)$, respectively.

First, we show that $p_\# : \pi_i(N) \to \pi_i(S^n)$ is an isomorphism for $i > 1$. Let $p : N \to S^n$ be the bundle projection. Since the base space S^n is a compact metric space, p is a fibration and there exists a homotopy exact sequence as follows;

$$\cdots \to \pi_i(F) \xrightarrow{i_\#} \pi_i(N) \xrightarrow{p_\#} \pi_i(S^n) \to \pi_{i-1}(F) \to \cdots,$$

where $i : F \to N$ is the inclusion. This sequence shows that $p_\# : \pi_i(N) \to \pi_i(S^n)$ is an isomorphism for $i \geq 2$.

Next, we claim that R induces isomorphisms $R_\# : \pi_i(N) \to \pi_i(N)$ for all positive integer i.

In the case $1 < i < n$, each i-th homotopy group of N is a trivial group and each $R_\# : \pi_i(N) \to \pi_i(N)$ is a trivial isomorphism.

For $i = n$, consider the homomorphism $R_\# : \pi_n(N) \to \pi_n(N)$ on the n-th homotopy groups. If $f : S^n \to N$ is a section of p, $p \circ f : S^n \to S^n$ is the identity map and thus it induces identity maps $(p \circ f)_* : H_n(S^n) \to H_n(S^n)$ and $(p \circ f)_\# : \pi_n(S^n) \to \pi_n(S^n)$. Hence, $f_\# : \pi_n(S^n) \to \pi_n(N)$ is actually an inverse isomorphism of $p_\#$ since $p_\#$ is an isomorphism. From the fact that $(p \circ f)_* = p_* \circ f_* = id$, we know that p_* is an epimorphism.

On the other side, the inclusion map $i : F \to N$ induces isomorphisms $i_\# : \pi_i(F) \to \pi_i(N)$ for $i \leq n - 1$. So Whitehead theorem guarantees that $i_* : H_i(F) \to H_i(N)$ is an isomorphism for $1 \leq i \leq n - 1$. Since $k \leq n - 1$, $H_n(N) \cong H^k(N) \cong H^k(F) \cong \mathbb{Z}$ and then $p_* : H_n(N) \to H_n(S^n)$ is an isomorphism by the hopfian property of $\mathbb{Z}$. Therefore f_* is the inverse isomorphism of p_*.

Consider the following commutative diagram;

$$
\begin{array}{ccc}
\pi_n(N) & \xrightarrow{R_\#} & \pi_n(N) \\
{\scriptstyle f_\#}\big\uparrow & & \big\downarrow{\scriptstyle p_\#} \\
\pi_n(S^n) & \xrightarrow{(p\circ R\circ f)_\#} & \pi_n(S^n) \\
{\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \text{Hurewicz map}} \\
H_n(S^n) & \xrightarrow{(p\circ R\circ f)_*} & H_n(S^n) \\
{\scriptstyle f_*}\big\downarrow & & \big\uparrow{\scriptstyle p_*} \\
H_n(N) & \xrightarrow{R_*} & H_n(N)
\end{array}
$$

Since R is a degree one map, $R_* : H_n(N) \to H_n(N)$ is an isomorphism and so is $(p \circ R \circ f)_* : H_n(S^n) \to H_{(}S^n)$. Applying Whitehead theorem for $(n-1)$-connected manifold S^n, it follows that $(p \circ R \circ f)_\# : \pi_n(S^n) \to \pi_n(S^n)$ is an isomorphism. Thus $R_\# : \pi_n(N) \to \pi_n(N)$ is an isomorphism.

Finally, let us prove that $R_\# : \pi_i(N) \to \pi_i(N)$ are isomorphisms for $i \geq n + 1$. Since $(p \circ R \circ f) : S^n \to S^n$ is a homology eqivalence and

S^n is simply connected, Whitehead theorem certifies the isomorphic property of $(p \circ R \circ f)_{\#} : \pi_i(S^n) \to \pi_i(S^n)$. Thus $R_{\#} : \pi_i(N) \to \pi_i(N)$ is an isomorphism and N is a Hopfian manifold.

COROLLARY 2.6. *Let N be a closed product bundle $S^n \times F$ over an n-sphere S^n having an aspherical closed k-manifold F as a fiber. Then N is a Hopfian manifold.*

3. Bundle spaces for codimension 2-fibrators

Recall that if N is a closed Hopfian n-manifold then an N-like decomposition map is an approximate fibration over its continuity set. Thus, using Theorem 2.2, we easily obtain the following results.

THEOREM 3.1. *If N is a closed bundle $F_1 \tilde{\times} F_2$ for aspherical manifolds F_1 and F_2 and $q : M \to B$ is a proper map from an arbitrary $(n+k)$-manifold M to a finite dimensional space B such that each point preimage $q^{-1}(b)$ is homotopic equivalent to N, $q : M \to B$ is an approximate fibration over its continuity set.*

Theorem 3.1 holds for each of all closed manifold described in Theorem 2.4 and 2.5, and Corollary 2.6.

In order for a Hopfian manifold N to be a codimension k-fibrator, N must equip the condition that the continuity set is the whole base set. The study on codimension 2-fibrators has many advantages compared with the other codimension and is necessary in the meaning that every codimension k-fibrator is a codimension $(k-1)$, moreover, at this time all known non-fibrators having no sphere as a Cartesian factor fail in codimension 2. In the case $k = 2$, the base space B is 2-manifold and $B \setminus C$ is locally finite in B [6], where C represents the continuty set of $q : M \to B$, and so we can localize to the situation which B is identical to E^2 and q is an approximate fibration over the complement of one point b.

From now on, we investigate closed manifolds with bundle structure to be codimension 2-fibrators. Call a finitely presented group H *hyperhopfian* if every endomorphism $\Psi : H \to H$ with $\Psi(H)$ normal and $H/\Psi(H)$ cyclic is an automorphism. Davermann showed that some conditions about a closed manifolds force to extend the continuity set to the whole set B.

LEMMA 3.2 [5]. *All closed, Hopfian manifolds with hyperhopfian fundamental group are codimension 2 fibrators.*

LEMMA 3.3 [5]. *Every closed, Hopfian manifold with a Hopfian fundamental group and nonzero Euler characteristic is a codimension 2 fibrator.*

A group H is said to be *residually finite* if for each $e_H \neq h \in H$, there exists a finite group A and a homomorphism $\phi : H \to A$ such that $\phi(h) \neq e_A$. Each finitely generated, residually finite group is Hopfian [15]. Also, a residually finite property is preserved under a semidirect product [13], that is, if H and K are residually finite groups then a semidirect product of H by K is residually finite.

THEOREM 3.4. *If each of F_1 and F_2 is an aspherical closed manifold with nonzero Euler characteristic and a residually finite fundamental group, then the bundle $F_1 \tilde{\times} F_2$ is a codimension 2-fibrator.*

Proof. A bundle projection over a closed manifold is a fibration and hence $\chi(F_1 \tilde{\times} F_2) = \chi(F_1)\chi(F_2) \neq 0$. From the homotopy exact sequence of three objects described in the proof of Theorem 2.2, we can see that the fundamental group of the aspherical manifold $F_1 \tilde{\times} F_2$ can be represented to be a semidirect product $\pi_1(F_2) \rtimes \pi_(F_1)$ of $\pi_1(F_2)$ and $\pi_(F_1)$. Since $\pi_1(F_1 \tilde{\times} F_2)$ is a finitely generated, residually finite group, it is hopfian. By Lemma 3.3, $F_1 \tilde{\times} F_2$ is a codimension 2-manifold.

COROLLARY 3.5. *If N is a bundle over S^n $(n \geq 2)$ having as a fiber a closed absolute retract manifold F, then it is a codimension 2 fibrator.*

Proof. An absolute retract is contractible and so F is aspherical. Hence S^n and F are compact metric spaces and thus the fiber N has a cross-section [18].

THEOREM 3.6. *If each of F_1 and F_2 is a closed surface with nonzero Euler characteristic, then the bundle N over F_1 with a fiber F_2 is a codimension 2-fibrator.*

Proof. Let $p : N \to F_1$ be a bundle projection and let us consider the following homotopy sequence;

$$\to \pi_2(F_2) \to \pi_2(N) \xrightarrow{p\#} \pi_2(F_1) \xrightarrow{g} \pi_1(F_2) \xrightarrow{i\#} \pi_1(N) \to \pi_1(F_1) \to 1$$

If both $\chi(F_1)$ and $\chi(F_2)$ are positive, F_1 and F_2 are simply connected spaces. From the above homotopy sequence, N is also simply connected which is a codimension 2-fibrator [4]. Otherwise, either F_1 or F_2 is asperical. By Theorem 2.4, the bundle N is a Hopfian manifold with hopfian fundamental group, which is a codimension 2-fibrator from Lemma 3.3.

Combining Theorem 2.5 with Lemma 3.2 and Lemma 3.3, the following results are obtained.

THEOREM 3.7. *If F be an aspherical closed manifold with nonzero Euler characteristic, then the bundle $S^n \tilde{\times} F$ having a section over n-sphere S^n is a codimension 2-fibrator.*

THEOREM 3.8. *If F is an aspherical closed manifold with hyperhopfian fundamental group then the bundle $S^n \tilde{\times} F$ having a section over n-sphere S^n is a codimension 2-fibrator.*

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