Pusan Kyŏngnam Math. J. 13(1997), No. 1, pp. 85–88

DIMENSIONS OF A POLYNOMIAL RING AND A POWER SERIES RING OVER A LOCALLY NOETHERIAN RING

CHUL JU HWANG AND GYU WHAN CHANG

1. Introduction

Let R be a commutative ring with identity and let $\{X_i\}_{i=1}^m$ be a set of indeterminates over R. It is well known that if R is a Noetherian ring, then $dimR[\{X_i\}_{i=1}^m] = dimR[|\{X_i\}_{i=1}^m|] = dimR+m$. In this paper, we will show that if R is a locally Noetherian ring, then $dimR[\{X_i\}_{i=1}^m] = dimR + m$, but $dimR[|\{X_i\}_{i=1}^m|]$ need not be $dimR + m$. Undefined notation and terminology will be essentially that of [1].

2. Main results

Throughout this section, R denotes a commutative ring with identity and $\{X_i\}_{i=1}^m$ is a set of indeterminates over R.

DEFINITION 1. A ring R is said to be locally Noetherian if for each maximal ideal M of R, R_M is Noetherian.

Let R be a locally Noetherian ring. If P is a prime ideal of R, there is a maximal ideal M of R with $P \subseteq M$. Since $R - M \subseteq R - P$, $R_P = (R_{R-M})_P = (R_M)_P$ and so R_P is Noetherian. So R is locally Noetherian if and only if R_P is Noetherian for each prime ideal P of R.

LEMMA 1. *If R is a Noetherian ring, then* $dimR[\{X_i\}_{i=1}^m] = dimR$ $[|\{X_i\}_{i=1}^m|] = dimR + m$.

Proof. Theorem 30.5 and Theorem 30.6 in [2].

Received Apr 16, 1997.

LEMMA 2. *If R is a locally Noetherian ring, then the polynomial ring $R[X]$ is also locally Noetherian.*

Proof. For a maximal ideal M of $R[X]$, let $P = M \cap R$. Then $R[X]_M = (R[X]_{(R-P)})_M = (R_P[X])_M$. Since R_P is Noetherian, $(R_P[X])_M$ is also Noetherian.

For a prime ideal P of R, the supremum of the lengths, taken over all strictly decreasing chains of prime ideals $P = P_0 \supset P_1 \supset P_2 \cdots \supset P_r$ starting from P, is called the height of P denoted by $ht(P)$.

LEMMA 3. *If R is a locally Noetherian domain, then $dim R[\{X_i\}_{i=1}^m] = dim R + m$.*

Proof. To avoid trivial cases, we can assume that $dim R = n < \infty$. By Lemma 2 we can assume that $m = 1$. It is clear that $n + 1 \leq dim R[X]$. To show that $dim R[X] \leq n + 1$, it is sufficient that for a prime ideal M of $R[X]$, $ht(M) \leq n + 1$. For a prime ideal M of $R[X]$, if $P = M \cap R$ then $ht(M) \leq ht(P[X]) + 1$. Since $R[X]_{P[X]} = (R_P[X])_{PR_P[X]}$, $ht(P[X]) = dim(R[X]_{P[X]}) = ht(PR_P[X]) \leq dim R_P [X] - 1 = (dim R_P + 1) - 1 = dim R_P = ht(P) \leq n$. Therefore, $ht(M) \leq ht(P[X]) + 1 \leq n + 1$.

LEMMA 4. *Let R be a ring with identity, then R is locally Noetherian if and only if for each nonzero ideal I of R, R/I is locally Noetherian.*

Proof. Suppose that R is locally Noetherian. Since a maximal ideal of R/I is of the form M/I where M is a maximal ideal of R containing I and $(R/I)_{M/I} \cong R_M/IR_M$, R/I is locally Noetherian. Conversely, for a maximal ideal M of R, each prime ideal of R_M is PR_M, where P is a prime ideal of R contained in M. Take a nonzero element $a \in P$, then R/aR is locally Noetherian and so $(R/aR)_{M/aR} \cong R_M/aR_M$ is Noetherian. Hence PR_M/aR_M is finitely generated and so is PR_M. Since PR_M is an arbitrary prime ideal of R_M, by Cohen's theorem R_M is Noetherian.

COROLLARY 1. *Let $\{X_i\}_{i=1}^m$ be a set of indeterminates over the ring R. Then $R[\{X_i\}_{i=1}^m]$ is locally Noetherian if and only if R is also locally Noetherian.*

THEOREM 1. *If R is a locally Noetherian ring, then* $dimR[\{X_i\}_{i=1}^m] = dimR + m$.

Proof. To avoid trivial cases, we can assume that $dimR = n < \infty$. Since $dimR[\{X_i\}_{i=1}^m] \geq n+m$, to prove the result it is enough to show that $dimR[\{X_i\}_{i=1}^m] \leq n+m$. By Lemma 2 we can assume that $m = 1$. For a mininmal prime ideal M_0 of $R[X]$, if $M_0 \cap R = P_0$ then $P_0[X]$ is a prime ideal of $R[X]$ contained in M_0. Since M_0 is a minimal prime ideal, $M_0 = P_0[X]$. Therefore $R[X]/M_0 \cong (R/P_0)[X]$. By Lemma 3 and Lemma 4 $dim(R[X]/M_0) = dim(R/P_0) + 1 \leq dimR + 1 = n + 1$. So $dimR[X] \leq n + 1$.

COROLLARY 2. *If R is an almost Dedekind domain, then* $dimR[X] = 2$.

We will give an example of an one dimensional locally Noetherian domain over which the power series ring has an infinite dimension.

DEFINITION 2. Let I be an ideal of the ring R. We shall call I an SFT-ideal if there exists a finitely generated ideal $B \subseteq I$ and a positive integer k such that $a^k \in B$ for each element $a \in I$.

LEMMA 5. *Suppose that M is a maximal ideal of a ring R such that M is not an SFT-ideal. Then* $ht(M[|X|]) = \infty$.

Proof. Theorem 21 in [1].

THEOREM 2. *If R is an almost Dedekind domain which is not Noetherian (Example 42.6 in [2]), then R is an one dimensional locally Noetherian domain and* $dimR[|X|] = \infty$.

Proof. Since R is an almost Dedekind domain R is a locally Noetherian domain. Since R is not Noetherian there is a maximal ideal M of R such that M is not finitely generated. Assume that M is a SFT-ideal. By definition there is a finitely generated ideal B and a positive integer k such that $a^k \in B$ for each element $a \in M$. It is clear that $\sqrt{B} = M$ and so B is M-primary. Since R_M is a DVR, $BR_M = (MR_M)^t = M^tR_M$ for some positive integer t. And $B = BR_M \cap R = M^tR_M \cap R = M^t$. Since B is finitely generated B is inveritible and so is M. Hence M is finitely generated, a contradiction. Therefore M is not a SFT-ideal and $ht(M[|X|]) = \infty$.

References

1. J. Brewer, *Power Series Over Commutative Rings*, Marcel Dekker, New York, 1981.
2. R. Gilmer,, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

Department of Mathematics
Pusan Women's University
Pusan 617-736, Korea

Department of Mathematics, Postech
Pohang 790-600, Korea