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NOTES ON HADAMARD PRODUCT OF CERTAIN MEROMORPHIC FUNCTIONS

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1. Introduction

Let $\sum$ be the class of functions of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_0 > 0, a_n \geq 0) \tag{1.1}$$

which are meromorphic in the punctured disk $D = \{z : 0 < |z| < 1\}$. A function $f(z) \in \sum$ is said to be meromorphic starlike of order α in D if it satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in D) \tag{1.2}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $\sum S_0^*(\alpha)$ the subclass of $\sum$ consisting of all such functions. A functions $f(z)$ in $\sum$ is said to be meromorphic convex of order α in D if it satisfies

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in D) \tag{1.3}$$

for some $\alpha(0 \leq \alpha < 1)$. We also denote by $\sum K_0(\alpha)$ the subclass of $\sum$ consisting of functions which are meromorphic convex of order α in D.

For functions $f_j(z)(j = 1, 2)$ defined by

$$f_j(z) = \frac{a_{0,j}}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \quad (a_{0,j} > 0, a_{n,j} \geq 0), \tag{1.4}$$

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we define the Hadamard product $f_1 * f_2(z)$ by

$$f_1 * f_2(z) = \frac{a_{0,1}a_{0,2}}{z} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}z^n. \tag{1.5}$$

In view of the previous definitions by Owa [4, 5, 6] and Kumar[1, 2], Mogra [3] has introduced the following class.

DEFINITION. A function $f(z) \in \sum$ is said to be a member of the class $\sum_k^*(\alpha)$ if and only if

$$\sum_{n=1}^{\infty} n^k(n+\alpha)a_n \leq (1-\alpha)a_0 \tag{1.6}$$

for some $\alpha(0 \leq \alpha < 1)$ and $k(k \geq 0)$.

REMARK 1. Note that $\sum_0^*(\alpha) \equiv \sum S_0^*(\alpha)$, and that $\sum_k^*(\alpha) \subset \sum_{k-1}^*(\alpha) \subset \cdots \subset \sum_2^*(\alpha) \subset \sum K_0(\alpha) \subset \sum S_0^*(\alpha)$.

For the above class $\sum_k^*(\alpha)$, Mogra [3] has proved

THEOREM A. *For each $j = 1, 2, \cdots, m$, let the functions $f_j(z)$ belong to the class $\sum S_0^*(\alpha_j)$, respectively. Then, the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\sum_{m-1}^*(\alpha^*)$, where $\alpha^* = \max_{1\leq j\leq m}\{\alpha_j\}$.*

THEOREM B. *For each $j = 1, 2, \cdots, m$, let the functions $f_j(z)$ belong to the class $\sum K_0(\alpha_j)$, respectively. Then, the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\sum_{2m-1}^*(\alpha^*)$, where $\alpha^* = \max_{1\leq j\leq m}\{\alpha_j\}$.*

THEOREM C. *For each $i = 1, 2, \cdots, m$, let the functions $f_i(z)$ belong to the class $\sum S_0^*(\alpha_i)$, respectively; and for each $j = 1, 2, \cdots, q$, let the functions $g_j(z)$ belong to the class $\sum K_0(\beta_j)$, respectively. Then, the Hadamard product $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ belongs to the class $\sum_{m+2q+1}^*(\gamma)$, where $\gamma = \max_{\substack{1\leq i\leq m \\ 1\leq j\leq q}}\{\alpha_i, \beta_j\}$.*

In this present paper, we prove the generalization theorems of the results by Mogra [3].

2. Generalization of theorems

We begin with the statement and the proof of our main result.

THEOREM 1. *If* $f_j(z) \in \sum_{k_j}^{*}(\alpha_j)$ *for each* $j = 1, 2, \cdots, m$, *then* $f_1 * f_2 * \cdots * f_m(z) \in \sum_{k}^{*}(\alpha)$, *where* $k = (\sum_{j=1}^{m} k_j) + m - 1$ *and* $\alpha = \max_{1 \le j \le m} \{\alpha_j\}$.

Proof. It is sufficient to prove that

$$\sum_{n=1}^{\infty}\left(n^k(n+\alpha)\prod_{j=1}^{m} a_{n,j}\right) \le (1-\alpha)\prod_{j=1}^{m} a_{0,j}. \tag{2.1}$$

Note that for $f_j(z) \in \sum_{k_j}^{*}(\alpha_j)$,

$$\sum_{n=1}^{\infty} n^{k_j}(n+\alpha_j)a_{n,j} \le (1-\alpha_j)a_{0,j} \quad (j = 1, 2, \cdots, m) \tag{2.2}$$

or

$$a_{n,j} \le \frac{(1-\alpha_j)a_{0,j}}{n^{k_j}(n+\alpha_j)} \le \frac{1}{n^{k_j+1}}a_{0,j} \quad (j = 1, 2, \cdots, m). \tag{2.3}$$

Further, we may assume that $\alpha = \max_{1 \le j \le m} \{\alpha_j\} = \alpha_m$. Then we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(n^k(n+\alpha)\prod_{j=1}^{m} a_{n,j}\right) \\
&\le \sum_{n=1}^{\infty}\left(n^k(n+\alpha_m)\Big(\prod_{j=1}^{m-1}\frac{1}{n^{k_j+1}}a_{0,j}\Big)a_{n,m}\right) \\
&= \Big(\prod_{j=1}^{m-1} a_{0,j}\Big)\left(\sum_{n=1}^{\infty} n^{k_m}(n+\alpha_m)a_{n,m}\right) \\
&\le (1-\alpha_m)\prod_{j=1}^{m} a_{0,j} \\
&= (1-\alpha)\prod_{j=1}^{m} a_{0,j}.
\end{aligned} \tag{2.4}$$

This completes the proof of theorem.

REMARK 2. Since $\sum_0^*(\alpha) \equiv \sum S_0^*(\alpha)$, letting $k_j = 0(j = 1, 2, \cdots, m)$ in Theorem 1, we have Theorem A by Mogra [3].

REMARK 3. If we take $k_j = 1(j = 1, 2, \cdots, m)$ in Theorem 1, then $k = (\sum_{j=1}^m k_j) + m - 1 = 2m - 1$. Therefore, taking $k_j = 1(j = 1, 2, \cdots, m)$, Theorem 1 leads to Theorem B by Mogra [3].

COROLLARY 1. *If $f_i(z) \in \sum_{k_i}^*(\alpha_i)$ for each $i = 1, 2, \cdots, m$, and if $g_j(z) \in \sum_{k_j'}^*(\beta_j)$ for each $j = 1, 2, \cdots, q$, then $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z) \in \sum_k^*(\gamma)$, where $k = \sum_{i=1}^m k_i + \sum_{j=1}^q k_j' + m + q - 1$ and $\gamma = \max_{\substack{1 \le i \le m \\ 1 \le j \le q}} \{\alpha_i, \beta_j\}$.*

The proof of the corollary is clear from Theorem 1.

REMARK 4. If we take $k_i = 0(i = 1, 2, \cdots, m)$ and $k_j' = 1(j = 1, 2, \cdots, q)$ in Corollary 1, then we have Theorem C by Mogra [3].

Next, we prove

THEOREM 2. *If $f_j(z) \in \sum_k^*(\alpha)$ for all $j = 1, 2, \cdots, m$, then $f_1 * f_2 * \cdots * f_m(z) \in \sum_{k+m-1}^*(\delta)$, where*

$$\delta = \delta(m, \alpha, \beta) = (1+\alpha)^m - \beta, \ \ 0 \le \alpha < 1, \ \ 1 \le \beta \le (1+\alpha)^m \tag{2.5}$$

and

$$(1-\alpha)^m + (1+\alpha)^m \le 1 + \beta. \tag{2.6}$$

Proof. It follows from $f_j(z) \in \sum_k^*(\alpha)(j = 1, 2, \cdots, m)$, that

$$\sum_{n=1}^{\infty} n^k(n+\alpha)a_{n,j} \le (1-\alpha)a_{0,j} \quad (j = 1, 2, \cdots, m). \tag{2.7}$$

From (2.7), we get

$$n^k(n+\alpha)a_{n,j} \le (1-\alpha)a_{0,j} \quad (j = 1, 2, \cdots, m) \tag{2.8}$$

for all n. From (2.8), we obtain

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(n^{k+m-1}(n+\delta)\prod_{j=1}^{m}a_{n,j}\right)\\
&=\sum_{n=1}^{\infty}\left(n^{k+m-1}(n+(1+\alpha)^{m}-\beta)\prod_{j=1}^{m}a_{n,j}\right)\\
&\leq\sum_{n=1}^{\infty}\left(n^{k+m-1}(n+\alpha)^{m}\prod_{j=1}^{m}a_{n,j}\right)\\
&\leq\sum_{n=1}^{\infty}n^{k}\left(n^{m-1}(n+\alpha)^{m-1}\prod_{j=1}^{m-1}a_{n,j}\right)(n+\alpha)a_{n,m}\\
&\leq\sum_{n=1}^{\infty}n^{k}\left(n^{m-1}\left(\frac{1-\alpha}{n^{k}}\right)^{m-1}\prod_{j=1}^{m-1}a_{n,j}\right)(n+\alpha)a_{n,m}\qquad(2.9)\\
&\leq\sum_{n=1}^{\infty}n^{k}\left((1-\alpha)^{m-1}\prod_{j=1}^{m-1}a_{0,j}\right)(n+\alpha)a_{n,m}\\
&\leq(1-\alpha)^{m}\prod_{j=1}^{m}a_{0,j}\\
&\leq(1-((1+\alpha)^{m}-\beta))\prod_{j=1}^{m}a_{0,j}\\
&=(1-\delta)\prod_{j=1}^{m}a_{0,j}
\end{aligned}
$$

where

$$(1-\alpha)^{m}+(1+\alpha)^{m}\leq 1+\beta,\ \ 0\leq\alpha<1\ \text{ and }\ 1\leq\beta\leq(1+\alpha)^{m}.$$

This implies that $f_1 * f_2 * \cdots * f_m(z) \in \sum_{k+m-1}^{*}(\delta)$.
Taking $k=0$ in Theorem 2, we have

COROLLARY 2. *If* $f_j(z) \in \sum S_0^*(\alpha)$ *for all* $j = 1, 2, \cdots, m$, *then* $f_1 * f_2 * \cdots * f_m(z) \in \sum_{m-1}^*(\delta)$, *where* δ *is given by (2.5), and* α *and* β *satisfy the condition in Theorem 2.*

Further, letting $k = 1$ in Theorem 2, we have

COROLLARY 3. *If* $f_j(z) \in \sum K_0(\alpha)$ *for all* $j = 1, 2, \cdots, m$, *then* $f_1 * f_2 * \cdots * f_m(z) \in \sum_m^*(\delta)$, *where* δ *is given by (2.5), and* α *and* β *satisfy the condition in Theorem 2.*

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