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GENERALIZED VECTOR VARIATIONAL INEQUALITIES AND GENERALIZED VECTOR COMPLEMENTARITY PROBLEMS ON H–SPACES

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–1. Introduction and preliminaries

Recently, Giannessi [5] has introduced the vector variational inequality (in short , VVI) in finite-dimensional Euclidean spaces. Since then Chen et al. [3, 4, 10, 12] have discussed and proved some existence theorems for vector variational inequalities and quasi-vector variational inequalities in Banach spaces. In [6, 7], Lee et al. established the existence theorem for the solution of (VVI) for multifunctions in reflexive Banach spaces.

On the other hand, classical complementarity problem has been considered as an equivalent form of variational inequality. This problem has wide-spread applications in economics and engineering together with variational inequality. Inspired and motivated by the applications of (VVI), we introduce in this paper a more general form of the (VVI) and vector complementarity problem (in short, VCP) corresponding to geneal vector variational inequalities (in short, H-$GVVI$) and general vector complementarity problem (in short, H-$GVCP$) on H-spaces, for which we can obtain nonconvex extensions of [6, 7, 8] Our main purpose of this paper is to investigate some existence theorems for the solutions of (H-$GVVI$) and (H-$GVCP$).

Let X and Y be two Banach spaces and D a nonempty closed convex subset of X. Let $T : X \to 2^{L(X,Y)}$ be a multifunction, where $L(X,Y)$ is the space of all continuous linear operators from X into Y. Let $\{C(x) : x \in D\}$ be a family of closed, pointed, and convex cones in Y such that $int\ C(x) \neq \emptyset$ for every $x \in D$, where int denotes the interior of a set.

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First we consider the following generalized vector variational inequality :

$(GVVI)$ Find $x_0 \in D$ such that for each $y \in D$, there exists an $s_0 \in T(x_0)$ satisfying $\langle s_0, y - g(x_0)\rangle \notin -int\ C(x_0)$, where $\langle s_0, y\rangle$ denotes the evaluation s_0 at y and $g : D \to D$ a mapping.

When g is an identity mapping, $(GVVI)$ reduces to the following generalized vector variational inequality $(GVVI)'$ considered by Lee et al. [7]:

$(GVVI)'$ Find $x_0 \in D$ such that for each $y \in D$, there exists an $s_0 \in T(x_0)$ satisfying $\langle s_0, y - x_0\rangle \notin -int\ C(x_0)$.

When T is an operator from X into $L(X,Y)$ and g is an identity mapping, $(GVVI)$ reduces to the following vector variational inequality (VVI) considered by Chen [3]:

(VVI) Find $x_0 \in D$ such that for each $y \in D$, $\langle T(x_0), y - x_0\rangle \notin -int\ C(x_0)$.

When for every $x \in D$, $C(x) = C$, where C is a closed, pointed, and convex cones in Y with $int\ C \neq \emptyset$, and g is identity mapping, $(GVVI)$ reduces to the following generalized vector variational inequality $(GVVI)''$ considerd by Lee et al. [6], and also (VVI) reduces to the following $(VVI)'$ considered by Chen et al. [4], Yang [12]:

$(GVVI)''$ Find $x_0 \in D$ such that for each $y \in D$, there exists an $s_0 \in T(x_0)$ such that $\langle s_0, y - x_0\rangle \notin -int\ C$.

$(VVI)'$ Find $x_0 \in D$ such that for each $y \in D$, $\langle T(x_0), y - x_0\rangle \notin -int\ C$.

When $Y = \mathbb{R}$, $X = \mathbb{R}^n, C(x) = \mathbb{R}_+$, then (VVI) collapses to the following classical scalar variational inequality (in short, VI):

(VI) Find $x_0 \in D$ such that $\langle f(x_0), y - x_0\rangle \geq 0$ for all $y \in D \subset \mathbb{R}^n$, where $f : D \to R^n$ is a given operator.

Now we recall some definitions and notations needed in this paper.

DEFINITION 1.1 [1]. An H-space is a pair $(X, \{\Gamma_A\})$, where X is a topological space, and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X, indexed by the finite subset of X such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

DEFINITION 1.2 [1]. Let $(X, \{\Gamma_A\})$ be an H-space, D be a nonempty subset of X.

(1) D is said to be H-convex if, for every finite subset $A \subset D$, it follows that $\Gamma_A \subset D$.
(2) D is said to be weakly H-convex if, for every finite subset $A \subset D$, $\Gamma_A \cap D$ is nonempty and contractible.
(3) A subset $K \subset X$ is said to be H-compact if , for every finite subset $A \subset X$, there exist a compact weakly H-convex subset $D \subset X$ such that $K \cup A \subset D$.

DEFINITION 1.3. In a given H-space $(X, \{\Gamma_A\})$, a map $F : X \to 2^X$ is called an H-KKM mapping if $\Gamma_A \subset \bigcup_{x \in A} F(x)$ for each finite subset A of X.

DEFINITION 1.4. A subset D of a topological space X is called compactly open (respectively, compactly closed) if for every compact set $K \subset X$, the set $D \cap K$ is open (respectively, closed) in K.

REMARK 1.5. It is easily shown that a closed-valued (respectively, open-valued) mapping $F:X \to 2^Y$ is compactly closed (respectively, compactly open). And a mapping $F:X \to 2^Y$ is compactly open on X if and only if a mapping $G:X \to 2^Y$ defined by, for every $x \in X$, $G(x) = Y \setminus F(x)$ is compactly closed on X.

DEFINITION 1.6 [10]. Let X and Y be two topological spaces. A mapping $F : X \to 2^Y$ is said to be transfer closed-valued on X if, for every $x \in X$, $y \notin F(x)$ implies that there exists a point $x' \in X$ such that $y \notin \overline{F(x')}$, the closure of $F(x')$.

DEFINITION 1.7 [10]. Let X and Y be two topological spaces. A mapping $F:X \to 2^Y$ is said to be transfer open-valued on X if, for every $x \in X$, $y \in F(x)$ implies that there exists a point $x' \in X$ such that $y \in int(F(x'))$, the interior of $F(x')$.

REMARK 1.8. It is easily proved that a closed-valued (respectively, open valued) mapping $F:X \to 2^Y$ is transfer closed-valued (respectively, transfer open-valued) by putting $x' = x$. Also a mapping $F:X \to 2^Y$ is transfer open-valued on X if and only if the mapping $G:X \to 2^Y$, defined by, for every $x \in X$, $G(X) = Y \setminus F(x)$, is transfer closed-valued on X.

DEFINITION 1.9. An H-spaces $(X, \{\Gamma_A\})$ is called an H-Banach space if X is a Banach space.

2. Main results

LEMMA 2.1 [11, 14]. *Let X be a nonempty set, Y a topological space and $G : X \to 2^Y$ a mapping. Then G is transfer closed-valued (respectively, transfer open-valued) if and only if*

$$\bigcap_{x\in X} G(x) = \bigcap_{x\in X} \overline{G(x)} \text{ (respectively, } \bigcup_{x\in X} G(x) = \bigcup_{x\in X} int(G(x)) \text{).}$$

Now we introduce the following H-KKM theoerm due to Bardaro and Ceppitelli [1].

THEOREM 2.2. *Let $(X, \{\Gamma_A\})$ be an H-space and $F : X \to 2^X$ an H-KKM mapping satisfying:*

(1) *for each $x \in X, F(x)$ is compactly closed,*
(2) *there is a compact set $L \subset X$ and an H-compact $K \subset X$ such that for each weakly H-convex set D with $K \subset D \subset X$, we have*

$$\bigcap_{x\in D} (F(x) \cap D) \subset L.$$

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

LEMMA 2.3. *Let $(X, \{\Gamma_A\})$ be an H-space and $F : X \to 2^X$ be an H-KKM mapping such that*

(1) *F is transfer closed-valued for each $x \in X$,*
(2) *there exists a compact subset L of X and an H-compact subset K of X such that for each weakly H-convex subset D of X with $K \subset D \subset X$, the following holds,*

$$\bigcap_{x\in D} (\overline{F(x)} \cap D) \subset L.$$

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

Proof. Define $\overline{F} : X \to 2^X$ by $\overline{F}(x) = \overline{F(x)}$ for each $x \in X$. Since F is an H-KKM mapping, $\overline{F}$ is also an H-KKM mapping with closed-valued. By Theorem 2.2, $\bigcap_{x\in X} \overline{F(x)} \neq \emptyset$. Since F is transfer closed-valued, by Lemma 2.1, we have $\bigcap_{x\in X} F(x) = \bigcap_{x\in X} \overline{F(x)} \neq \emptyset$. This completes the proof.

We begin with the following Theorem 2.4 which is the existence theorem for a generalized vector variational inequality on H-Banach spaces.

In the sequel, $F(A) = \bigcup\{F(x) : x \in A\}$ and a multifunction $F : X \to 2^Y$ is compact provided $F(X)$ is contained in a compact subset of Y.

THEOREM 2.4. *Let* $(X, \{\Gamma_A\})$ *be an H-Banach space,* Y *be a Banach space,* $\{C(x) : x \in X\}$ *be a family of closed, pointed and convex cone in* Y *such that* $int\ C(x) \neq \emptyset$ *for every* $x \in X$*, and a multifunction* $W : X \to 2^Y$*, defined by* $W(x) = Y \setminus int\ C(x)$ *for any* $x \in X$*, be closed. Assume that:*

(1) $T : X \to 2^{L(X,Y)}$ *is a upper semi-continuous and compact multifunction, and* $g : X \to X$ *is continuous;*
(2) *for each* $y \in X$*, there exists* $s \in T(y)$ *such that* $\langle s, y - g(y)\rangle \notin -int\ C(y)$*;*
(3) *for each* $y \in X$*,* $B_y = \{x \in X : \langle s, x - g(y)\rangle \in -int\ C(x)$ *for any* $s \in T(x)\}$ *is H-convex;*
(4) *there exist a compact set* $L \subset X$ *and an H-compact set* $K \subset X$ *such that, for every weakly H-compact set* D *with* $K \subset D \subset X$*, we have*

$$\bigcap_{y\in D} \{x \in D : \exists s \in T(x)\ s.t.\ \langle s, y - g(x)\rangle \notin -int\ C(x)\} \subset L.$$

Then the following generalized vector variational inequality on H-space (H-GVVI) is solvable.

(H-GVVI) Find $x \in X$ *such that for each* $y \in X$*, there exists an* $s \in T(x)$ *satisfying* $\langle s, y - g(x)\rangle \notin -int\ C(x)$

Proof. Define a multifunction $F : X \to 2^X$ by

$$F(y) = \{x \in X : \langle s, y - g(x)\rangle \notin -int\ C(x) \text{ for some } s \in T(x)\}$$

for $y \in X$. Then F is an H-KKM mapping on X. In fact, suppose that F is not an H-KKM mapping, then there exists a finite set $A \subset X$ such that $\Gamma_A \not\subset \bigcup_{x\in A} F(x)$. Thus there exists $z \in \Gamma_A$ such that $z \notin F(x)$ for each $x \in A$ and hence for any $s \in T(z)$ and $x \in A$, we have $\langle s, x - g(z)\rangle \in -int\ C(z)$.

Since B_z is H-convex by (3) and $A \subset B_z$, $\Gamma_A \subset B_z$. Therefore $z \in B_z$, and for any $s \in T(z)$, $\langle s, z - g(z)\rangle \in -int\ C(z)$, which contradicts the assumption (2). Thus F is an H-KKM mapping.

Now, we will prove that for each $y \in X$, $F(y)$ is closed. Indeed, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $F(y)$ such that $x_n \to x_*$ for any fixed $y \in X$. Since $x_n \in F(y)$ for all n, there exists an $s_n \in T(x_n)$ such that $\langle s_n, y - g(x_n)\rangle \in W(x_n) = Y \setminus (-int\ C(x_n))$ for all n. From (1), we can assume that there exists $s_* \in L(X,Y)$ such that $s_n \to s_*$ and $s_* \in T(x_*)$. Since g is continuous, then $g(x_n) \to g(x_*)$. Since $\langle \cdot,\cdot \rangle$ is continuous, $\langle s_n, y - g(x_n)\rangle \to \langle s_*, y - g(x_*)\rangle$. Since $\langle s_n, y - g(x_n)\rangle \in W(x_n)$ and W is closed, we have $\langle s_*, y - g(x_*)\rangle \in W(x_*)$, i.e. , $x_* \in F(y)$. Hence $F(y)$ is closed; it is natural that $F(y)$ is transfer closed. It is easy to see that the present assumption (4) is the same as the assumption (2) of Lemma 2.3. Thus by Lemma 2.3, $\bigcap_{y\in X} F(y) \neq \emptyset$, i.e. , there exists $x \in X$ such that for each $y \in X$, there exists an $s \in T(x)$ satisfying $\langle s, y - g(x)\rangle \notin -int\ C(x)$. This completes the proof.

THEOREM 2.5. *Let $(X, \{\Gamma_A\})$ be an H-Banach space, Y be Banach space, $\{C(x) : x \in X\}$ be a family of closed, pointed and convex cone in Y such that $int\ C(x) \neq \emptyset$ for every $x \in X$, and $T : X \to 2^{L(X,Y)}$ be multifunction. If $x \in X$ is a solution of the above (H-GVVI), then $x \in X$ is a solution of the following generalized vector complementarity problem on H-space (H-GVCP).*

(H-GVCP) Find $x \in X$ such that there exists an $s \in T(x)$ satisfying $\langle s, g(x)\rangle \notin int\ C(x)$, where $s \in C^* = \{q \in L(X,Y) : \langle q, x\rangle \notin -int\ C(x),\ x \in X\}$.

Proof. Let x solve H-GVVI. Also letting $y = w + g(x)$ for each $w \in X$, we have $\langle s, w\rangle \notin int\ C(x)$ for each $w \in X$. Hence we have that there exists $s \in T(x)$ such that $\langle s, g(x)\rangle \notin int\ C(x)$ and such that $\langle s, w\rangle \notin -\text{int}\ C(x)$ for each $w \in X$. Therefore $s \in C^*$. This implies that x solves $H\text{-}GVCP$.

By Theorem 2.4 and Theorem 2.5, we can obtain the following Theorem 2.6 which is the existence theorem for $(H\text{-}GCP)$.

Theorem 2.6. *Let $(X, \{\Gamma_A\})$ be an H-Banach space, Y be a Banach space, $\{C(x)\}$ be a family of closed, pointed and convex cone in Y such that $int\ C(x) \neq \emptyset$ for every $x \in X$, and a multifunction $W : X \to 2^Y$, defined by $W(x) = Y \setminus int\ C(x)$ for any $x \in X$, be closed. Assume that:*

(1) *$T : X \to 2^{L(X,Y)}$ is a upper semi-continuous and compact multifunction, and $g : X \to X$ is continuous;*
(2) *for each $y \in X$, there exists $s \in T(y)$ such that $\langle s, y - g(y)\rangle \notin -int\ C(y)$;*
(3) *for each $y \in X$, $B_y = \{x \in X : \langle s, x - g(y)\rangle \in -int\ C(y)$ for $s \in T(y)\}$ is H-convex;*
(4) *there exists a compact set $L \subset X$ and an H-compact set $K \subset X$ such that, for every weakly H-compact set D with $K \subset D \subset X$, we have*

$$\bigcap_{y\in D} \{x \in D : \exists s \in T(x)\ s.t.\ \langle s, y - g(x)\rangle \notin -int\ C(x)\} \subset L.$$

Then (H-GVCP) is solvable.

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