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INTERPOLATION SPACES GENERATED BY C_0–SEMIGROUP OPERATOR

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1. Introduction

In this paper we deal with an interpolation method between the initial Banach space and the domain of the infinitesimal generator A of the C_0-semigroup $T(t)$ and the fundamental results of the corresponding theorems in the new setting. The objects are obtained by the development of an interpolation theory between Banach spaces X and Y, which is denoted by $(X,Y)_{\theta,p}$, in particular by the J- and K-methods as in Butzer and Berens [1] and [2]. We will make easier some proofs of the fact that

$$(D(A),X)_{\theta,p} = \{x \in X : \int_0^t (t^{\theta-1}||T(t)x - x||)^p \frac{dt}{t} < \infty\}.$$

It is mainly on the role of interolation spaces in the study of C_0-semigroup of operators. In forth coming paper, we will deal with interpolation spaces between the initial Banach spaces and the domain of the infinitesimal generator of an analytic semigroup.

2. Preliminaries

Let X and Y be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}$ such that the embedding mapping of both X and Y in $\mathcal{X}$ is continuous. Let $X \cap Y$ be a dense subspace in both X and Y. For $1 < p < \infty$, we denote by $L_*^p(X)$ the Banach space of all functions $t \to u(t)$, $t \in (0,\infty)$ and $u(t) \in X$, for which the mapping $t \to u(t)$ is strongly measurable with respect to the measure dt/t and the norm $||u||_{L_*^p(X)}$ is finite, where

$$||u||_{L_*^p(X)} = \{\int_0^\infty ||u(t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

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For $0 < \theta < 1$, set

$$||t^\theta u||_{L^p_*(X)} = \{\int_0^\infty ||t^\theta u(t)||^p_X \frac{dt}{t}\}^{\frac{1}{p}},$$
$$||t^\theta u'||_{L^p_*(Y)} = \{\int_0^\infty ||t^\theta u'(t)||^p_Y \frac{dt}{t}\}^{\frac{1}{p}}.$$

We now introduce a Banach space

$$V = \{u : ||t^\theta u||_{L^p_*(X)} < \infty, \quad ||t^\theta u'||_{L^p_*(Y)} < \infty\}$$

with norm

$$||u||_V = ||t^\theta u||_{L^p_*(X)} + ||t^\theta u'||_{L^p_*(Y)}.$$

DEFINITION 2.1. We define $(X, Y)_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$, to be the space of all elements $u(0)$ where $u \in V$, that is,

$$(X, Y)_{\theta,p} = \{u(0) : u \in V\}.$$

LEMMA 2.1. *Let $0 < \theta < 1$, $1 < p < \infty$ and $\phi(t) \ge 0$ almost everywhere. Then*

$$\{\int_0^\infty (t^{\theta-1}\int_0^t \phi(s)ds)^p \frac{dt}{t}\}^{\frac{1}{p}} \le \frac{1}{1-\theta}\{\int_0^\infty (t^\theta \phi(t))^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

Proof. Let $0 < \epsilon < N < \infty$. Then

$$\begin{aligned}
&\int_\epsilon^N (t^{\theta-1}\int_0^t \phi(s)ds)^p \frac{dt}{t}\\
&= \int_\epsilon^N t^{(\theta-1)p-1}(\int_0^t \phi(s))ds)^p dt\\
&= [\frac{t^{(\theta-1)p}}{(\theta-1)p}(\int_0^t \phi(s)ds)^p]_\epsilon^N - \int_\epsilon^N \frac{t^{(\theta-1)p}}{(\theta-1)p} p\phi(t)(\int_0^t \phi(s)ds)^{p-1}dt\\
&\le \frac{\epsilon^{(\theta-1)p}}{(1-\theta)p}(\int_0^\epsilon \phi(s)ds)^p + \frac{1}{1-\theta}\int_\epsilon^N t^{(\theta-1)p}\phi(t)(\int_0^t \phi(s)ds)^{p-1}dt.
\end{aligned}$$

Since

$$\begin{aligned}\int_0^\epsilon \phi(s)ds &= \int_0^\epsilon s^{1-\theta}s^\theta\phi(s)\frac{ds}{s}\\ &\le \{\int_0^\epsilon s^{(1-\theta)p'}\frac{ds}{s}\}^{\frac{1}{p'}}\{\int_0^\epsilon (s^\theta\phi(s))^p\frac{ds}{s}\}^{\frac{1}{p}}\\ &= \{\frac{\epsilon^{(1-\theta)p'}}{(1-\theta)p'}\}^{\frac{1}{p'}}\{\int_0^\epsilon (s^\theta\phi(s))^p\frac{ds}{s}\}^{\frac{1}{p}},\end{aligned}$$

we see that

$$\epsilon^{\theta-1}(\int_0^\epsilon \phi(s)ds) \le (\frac{1}{(1-\theta)p'})^{\frac{1}{p'}}\{\int_0^\epsilon (s^\theta\phi(s))^p\frac{ds}{s}\}^{\frac{1}{p}}$$

tends to zero as ϵ tends to zero. If $\epsilon \to 0$ and $N \to \infty$, then we have

$$\begin{aligned}&\int_0^\infty (t^{\theta-1}\int_0^t \phi(s)ds)^p\frac{dt}{t}\\ &\le \frac{1}{1-\theta}\{\int_0^\infty t^{(\theta-1)p}\phi(t)(\int_0^t \phi(s)ds)^{p-1}dt\\ &= \frac{1}{1-\theta}\int_0^\infty t^{(\theta-1)(p-1)+\theta}\phi(t)(\int_0^t \phi(s)ds)^{p-1}\frac{dt}{t}\\ &= \frac{1}{1-\theta}\int_0^\infty t^\theta\phi(t)(t^{\theta-1}\int_0^t \phi(s)ds)^{p-1}\frac{dt}{t}\\ &\le \frac{1}{1-\theta}\{\int_0^\infty (t^\theta\phi(t))^p\frac{dt}{t}\}^{\frac{1}{p}}\{\int_0^\infty (t^{\theta-1}\int_0^t \phi(s)ds)^p\frac{dt}{t}\}^{1-\frac{1}{p}}.\end{aligned}$$

Hence the proof is complete.

LEMMA 2.2. *Let $\theta < 2, 1 < p < \infty$ and $\phi(t) \ge 0$ almost everywhere. Then*

$$\{\int_0^\infty (t^{\theta-2}\int_0^t s^{1-\theta}\phi(s)ds)^p\frac{dt}{t}\}^{\frac{1}{p}} \le \frac{1}{2-\theta}\{\int_0^\infty \phi(t)^p\frac{dt}{t}\}^{\frac{1}{p}}.$$

Proof. Let $0 < \epsilon < N < \infty$. Then

$$\begin{aligned}
&\int_\epsilon^N (t^{\theta-2}\int_0^t s^{1-\theta}\phi(s)ds)^p \frac{dt}{t}\\
&= \int_\epsilon^N t^{(\theta-2)p-1}(\int_0^t s^{1-\theta}\phi(s))ds)^p dt\\
&= [\frac{t^{(\theta-2)p}}{(\theta-2)p}(\int_0^t s^{1-\theta}\phi(s)ds)^p]_\epsilon^N\\
&\quad - \int_\epsilon^N \frac{t^{(\theta-2)p}}{\theta-2} t^{1-\theta}\phi(t)(\int_0^t s^{1-\theta}\phi(s)ds)^{p-1}dt\\
&\le \frac{\epsilon^{(\theta-2)p}}{(2-\theta)p}(\int_0^\epsilon s^{1-\theta}\phi(s)ds)^p\\
&\quad + \frac{1}{2-\theta}\int_\epsilon^N t^{(\theta-2)p+1-\theta}\phi(t)(\int_0^t s^{1-\theta}\phi(s)ds)^{p-1}dt.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^\epsilon s^{1-\theta}\phi(s)ds &= \int_0^\epsilon s^{2-\theta}\phi(s)\frac{ds}{s}\\
&\le \{\int_0^\epsilon s^{(2-\theta)p'}\frac{ds}{s}\}^{\frac{1}{p'}}\{\int_0^\epsilon \phi(s)^p\frac{ds}{s}\}^{\frac{1}{p}}\\
&= \{\frac{\epsilon^{(2-\theta)p'}}{(2-\theta)p'}\}^{\frac{1}{p'}}\{\int_0^\epsilon \phi(s)^p\frac{ds}{s}\}^{\frac{1}{p}},
\end{aligned}$$

we also see that

$$\epsilon^{\theta-2}(\int_0^\epsilon s^{1-\theta}\phi(s)ds) \le (\frac{1}{(2-\theta)p'})^{\frac{1}{p'}}\{\int_0^\epsilon \phi(s)^p\frac{ds}{s}\}^{\frac{1}{p}}$$

tends to zero as ϵ tends to zero. If $\epsilon \to 0$ and $N \to \infty$, then we have

$$\begin{aligned}
&\int_0^\infty (t^{\theta-2}\int_0^t s^{1-\theta}\phi(s)ds)^p\frac{dt}{t}\\
&\le \frac{1}{2-\theta}\int_0^\infty (t^{(\theta-2)p+1-\theta}\phi(t)(\int_0^t s^{1-\theta}\phi(s)ds)^{p-1}dt\\
&= \frac{1}{2-\theta}\int_0^\infty \phi(t)(t^{\theta-2}\int_0^t s^{1-\theta}\phi(s)ds)^{p-1}\frac{dt}{t}\\
&\le \frac{1}{2-\theta}\{\int_0^\infty \phi(t)^p\frac{dt}{t}\}^{\frac{1}{p}}\{\int_0^\infty (t^{\theta-2}\int_0^t s^{1-\theta}\phi(s)ds)^p\frac{dt}{t}\}^{1-\frac{1}{p}},
\end{aligned}$$

and hence the proof is complete.

3. Main result

Let X be a Banach space with norm $||\cdot||$ and $T(t)$ be a C_0-semigroup with infinitesimal generator A. Then its domain $D(A)$ is a Banach space with the graph norm $||x||_{D(A)} = ||Ax|| + ||x||$.

The following result is the various possible approach to the theory of interpolation spaces and some of its many applications to mathematical analysis.

THEOREM 3.1. *Let $0 < \theta < 1$, $1 < p < \infty$. Then*

$$(D(A), X)_{\theta,p} = \{x \in X : \int_0^t (t^{\theta-1}||T(t)x - x||)^p \frac{dt}{t} < \infty\}.$$

Proof. Let $x \in (D(A), X)_{\theta,p}$. Then there exists $u \in V$ such that $x = u(0)$,

$$||t^\theta u||_{L^p_*(D(A))} < \infty, \quad ||t^\theta u'||_{L^p_*(X)} < \infty.$$

Put $u'(t) - Au(t) = f(t)$. Then $||t^\theta f||_{L^p_*(X)} < \infty$ and

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

Since

$$\begin{aligned} T(t)x - x &= u(t) - \int_0^t T(t-s)f(s)ds - x \\ &= \int_0^t u'(s)ds - \int_0^t T(t-s)f(s)ds, \end{aligned}$$

it holds

$$||T(t)x - x|| \le \int_0^t ||u'(s)||ds + M\int_0^t ||f(s)||ds.$$

From Lemma 2.1 it follows that

$$\begin{aligned}
&\{\int_0^\infty (t^{\theta-1}\|T(t)x - x\|)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \{\int_0^\infty (t^{\theta-1}\int_0^t \|u'(s)\|ds)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + M\{\int_0^\infty (t^{\theta-1}\int_0^t \|f(s))\|ds)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \frac{1}{1-\theta}\{\int_0^\infty (t^{\theta}\|u'(t)\|)^p \frac{dt}{t}\}^{\frac{1}{p}} + \frac{M}{1-\theta}\{\int_0^\infty t^{\theta}\|f(t)\|)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&= \frac{1}{1-\theta}\|t^\theta u'\|_{L^p_*(X)} + \frac{M}{1-\theta}\|t^\theta f\|_{L^p_*(X)} < \infty.
\end{aligned}$$

On the other hand, let

$$\int_0^t (t^{\theta-1}\|T(t)x - x\|)^p \frac{dt}{t} < \infty.$$

Put

$$v(t) = \frac{1}{t}\int_0^t T(s)xds.$$

Then

$$\begin{aligned}
v'(t) &= \frac{1}{t}T(t)x - \frac{1}{t^2}\int_0^t T(s)xds \\
&= \frac{1}{t}(T(t)x - x) - \frac{1}{t^2}\int_0^t (T(s)x - x)ds
\end{aligned}$$

and

$$Av(t) = \frac{1}{t}(T(t)x - x).$$

Here, we remark that since A is closed we have

$$A\int_0^t T(s)xds = T(t)x - x$$

for every $x \in X$. Thus it holds

$$v'(t) = Av(t) - w(t), \quad w(t) = \frac{1}{t^2}\int_0^t (T(s) - x)ds$$

and
$$\int_0^\infty \|t^\theta Av(t)\|^p \frac{dt}{t} = \int_0^\infty (t^{\theta-1}\|T(t)x - x\|)^p \frac{dt}{t} < \infty.$$
From Lemma 2.2 and
$$\begin{aligned}
&\{\int_0^\infty \|t^\theta w(t)\|^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&= \{\int_0^\infty \|t^{\theta-2} \int_0^t (T(s)x - x)ds\|^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \{\int_0^\infty (t^{\theta-2} \int_0^t s^{1-\theta} s^{\theta-1}\|T(s)x - x\|)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \frac{1}{2-\theta}\{\int_0^\infty (t^{\theta-1}\|T(t)x - x\|)^p \frac{dt}{t}\}^{\frac{1}{p}} < \infty,
\end{aligned}$$

it follows that

$$\|t^\theta v'\|_{L_*^p(X)} \le \|t^\theta Av\|_{L_*^p(X)} + \|t^\theta w\|_{L_*^p(X)} < \infty. \tag{3.1}$$

Choose $q \in C_0^1([0,\infty))$ such that $q(0) = 1$, $0 \le q(t) \le 1$ and we can put $u(t) = q(t)v(t)$ safisfying $u(0) = x$. Then
$$u'(x) = q(t)v'(t) + q'(t)v(t)$$

and we can estimate that

$$\begin{aligned}
\|t^\theta u'\|_{L_*^p(X)} &\le \|t^\theta qv'\|_{L_*^p(X)} + \|t^\theta q'v\|_{L_*^p(X)} \\
&\le \|t^\theta v'\|_{L_*^p(X)} + \|t^\theta q'v\|_{L_*^p(X)}
\end{aligned} \tag{3.2}$$

Here, the eatimate of the second term of mentioned above is

$$\begin{aligned}
\|t^\theta q'v\|_{L_*^p(X)} &= \{\int_0^\infty \|t^\theta q'(t)v(t)\|^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&= \{\int_0^\infty t^{\theta p-1}|q'(t)|^p\|v(t)\|^p dt\}^{\frac{1}{p}} \\
&\le \max_{q'(t)\neq 0} |q'(t)|\{\int_0^\infty t^{\theta p-1}\|v(t)\|^p dt\}^{\frac{1}{p}} \\
&\le \max_{q'(t)\neq 0} |q'(t)|\|t^\theta v\|_{L_*^p(X)} \\
&< \infty.
\end{aligned}$$

It is easily known that

$$||t^\theta v||_{L^p_*(X)} < \infty.$$

Hence we have that from (3.1)

$$(3.3) \qquad ||t^\theta u'||_{L^p_*(X)} < \infty.$$

It also holds that

$$(3.4) \qquad \begin{aligned} ||t^\theta u||_{L^p_*(D(A))} &\le ||t^\theta Au||_{L^p_*(X)} + ||t^\theta u||_{L^p_*(X)} \\ &\le ||t^\theta Av||_{L^p_*(X)} + ||t^\theta v||_{L^p_*(X)} \\ &< \infty. \end{aligned}$$

From (3.3) and (3.4) we conclude that

$$x \in (D(A), X)_{\theta,p}.$$

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