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KIVENTIDIS TYPE FIXED POINT THEOREMS

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I. Introduction

Let (X, d) be a metric space and S, T be mappings from X into itself. Then a point $x \in X$ such that $x = Sx = Tx$ is called a common fixed point of S and T. If we put $S = I_X$ (: the identity mapping on X), i.e., $x = Tx$, then the point x is called a fixed point of T.

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of such a fixed point, which is called Banach's fixed point theorem or the Banach contraction principle. This theorem is also applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other applied mathematics and many authors extended, generalized and improved Banach's fixed point theorem in different ways. In [6], Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible but the converses are not necessarily true ([6], [19]). Several authors proved some common fixed point theorems for commuting, weakly commuting and compatible mappings ([1]-[4], [6]-[8], [10], [16]-[20]). Recently, Jungck, Murthy and Cho [9] defined the concept of compatible mappings of type (A) which is equivalent to the concept of compatible mappings under some conditions and proved a common fixed point theorem for compatible mappings of type (A) in metric spaces and Banach spaces ([9], [12]). In [13], Pathak and Khan introduced the concept of compatible mappings of type (B) and compared these mappings with compatible mappings and compatible mappings of type (A) in normed linear spaces. In the sequel, they derived some relations between these mappings and proved a common fixed point

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theorems of Gregus type for compatible mappings of type (B) in Banach spaces.

In [11], T. Kiventidis proved the following:

THEOREM A. *Let T be a mapping from a complete metric space (X, d) into itself satisfying the following condition:*

$$d(Tx, Ty) \leq d(x, y) - w(d(x, y)) \tag{1.1}$$

for all $x, y \in X$, where $w : R^+ = [0, \infty) \to R^+$ is a continuous function such that $0 < w(r) < r$ for all $r \in R^+ - \{0\}$. Then T has a unique fixed point in X.

In [15], Ray extended Theorem A by using the concept of commuting mappings in a metric space as follows:

THEOREM B. *Let A, B and T be mappings from a complete metric space (X, d) into itself such that $A(X) \cup B(X) \subset T(X)$, $AT = TA$, T is continuous and*

$$d(Ax, By) \leq d(Tx, Ty) - w(d(Tx, Ty)) \tag{1.2}$$

for all $x, y \in X$, where $w : R^+ \to R^+$ is continuous function such that $0 < w(r) < r$ for all $r : R^+ - \{0\}$. Then A, B and T have a unique common fixed point in X.

Very recently, in [17], Rhoades, Tiwary and Singh extended also Theorem A by using the concept of compatible mappings (cf. Definition 2.1) in a metric space as follows:

THEOREM C. *Let A and B be continuous mappings from a complete metric space (X, d) into itself. Then A and B have a common fixed point in X if and only if there exists a continuous mapping $T : X \to f(X) \cap g(X)$ such that the pairs $\{A, T\}$ and $\{B, T\}$ are compatible,*

$$\begin{aligned} d(Tx, Ty) \leq \max\{&d(Tx, Ax), d(Ty, By), d(Ax, By), \\ &\frac{1}{2}d(Tx, By) + d(Ty, Ax)]\} \\ &- w(\max\{d(Tx, Ax), d(Ty, By), d(Ax, By), \\ &\frac{1}{2}d(Tx, By) + d(Ty, Ax)]\}) \end{aligned} \tag{1.3}$$

for all $x, y \in X$, where $w : R^+ \to R^+$ is continuous function such that $0 < w(r) < r$ for all $r \in R^+ - \{0\}$. Indeed, A, B and T have a unique common fixed point in X.

In this paper, by using the concept of compatible mappings of type (B) in metric spaces, we give some common fixed point theorems for four compatible mappings of type (B) satisfying the more general contractive mappings of the Kiventidis type in metric spaces. Finally, we give also some convergence theorems for self-mappings in metric spaces satisfying some conditions. Our main results extend, generalize and improve Theorems A, B, C and many others for commuting, weakly commuting and compatible mappings.

II. Compatible mappings of type (B)

In this section, we introduce the concept of compatible mappings of type (B) in metric spaces and show that, under some conditions, these mappings are equivalent to compatible mappings, compatible mappings of types (A) and (B) in metric spaces. Now, we state some definitions, examples and propositions for our main results:

DEFINITION 2.1 [6]. Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

DEFINITION 2.2 [9]. Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be *compatible of type* (A) if

$$\lim_{n\to\infty} d(TSx_n, SSx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(STx_n, TTx_n) = 0$$

when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

DEFINITION 2.3 [13]. Let S and T be mappings from a metric space (X,d) into itself. The mappings S and T are said to be *compatible of type* (B) if

$$\lim_{n\to\infty} d(STx_n,TTx_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(STx_n,St) + \lim_{n\to\infty} d(St,SSx_n)],$$
$$\lim_{n\to\infty} d(TSx_n,SSx_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(TSx_n,Tt) + \lim_{n\to\infty} d(Tt,TTx_n)]$$

when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

We give some properties and relations on compatible mappings and compatible mappings of types (A) and (B) in metric spaces.

PROPOSITION 2.1 [9]. *Let S and T be continuous mappings of a metric space (X,d) into itself. If S and T are compatible, then they are compatible of type (A).*

PROPOSITION 2.2 [9]. *Let S and T be compatible mappings of type (A) from a metric space (X,d) into itself. If one of S and T is continuous, then S and T are compatible.*

From Propositions 2.1 and 2.2, we have the following:

PROPOSITION 2.3 [9]. *Let S and T be continuous mappings from a metric space (X,d) into itself. Then S and T are compatible if and only if they are compatible of type (A).*

By suitable examples, Jungck, Murthy and Cho [9] have shown that Proposition 2.3 is not true if S and T are not continuous.

PROPOSITION 2.4 [13]. *Let S and T be compatible mappings of type (A) from a metric space (X,d) into itself. Then S and T are compatible mappings of type (B).*

PROPOSITION 2.5 [13]. *Let S and T be continuous mappings of a metric space (X, d) into itself. If S and T are compatible of type (B), then they are compatible of type (A).*

PROPOSITION 2.6 [13]. *Let S and T be continuous mappings of a metric space (X,d) into itself. If S and T are compatible mappings of type (B), then they are compatible.*

Proposition 2.7 [13]. *Let S and T be continuous mappings from a metric space (X,d) into itself. If S and T are compatible, then they are compatible of type (B).*

From Propositions 2.4∼2.7, we have the following:

Proposition 2.8 [13]. *Let S and T be continuous mappings of a metric space (X,d) into itself. Then*

(1) *S and T are compatible if and only if they are compatible of type (B).*

(2) *S and T are compatible of type (A) if and only if they are compatible of type (B).*

Pathak and Khan [13] gave some examples that Proposition 2.8 is not true if S and T are not continuous.

Proposition 2.9 [13]. *Let S and T be compatible mappings of type (B) from a metric space (X,d) into itself. If $St = Tt$ for some $t \in X$, then $STt = TSt = SSt = TTt$.*

Proposition 2.10 [13]. *Let S and T be compatible mappings of type (B) from a metric space (X,d) into itself. If $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$, then*

(1) $\lim_{n\to\infty} TTx_n = St$ *if S is continuous at t,*

(2) $\lim_{n\to\infty} SSx_n = Tt$ *if T is continuous at t,*

(3) *$STt = TSt$ and $St = Tt$ if S and T are continuous at t.*

The following examples which give some relations between compatible mappings of types (A) and (B) are given in [14]:

Example 2.1. Let $X = [0,\infty)$ be a metric space with the Euclidean metric space $d(x,y) = |x-y|$. Define the mappings S, $T : X \to X$ by

$$S(x) = \begin{cases} \frac{1}{2} + x & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x \in (\frac{1}{2}, \infty), \end{cases} \qquad T(x) = \begin{cases} \frac{1}{2} - x & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x \in (\frac{1}{2}, \infty), \end{cases}$$

respectively. Then S and T are compatible, but they are neither compatible of type (A) nor compatible of type (B).

EXAMPLE 2.2. Let $X = [0, \infty)$ be a metric space with the Euclidean metric $d(x, y) = |x - y|$. Define S and $T : X \to X$ by

$$S(x) = \begin{cases} \frac{1}{2} + x & \text{if } x \in [0, \frac{1}{2}) \\ 2 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x \in (\frac{1}{2}, \infty), \end{cases} \qquad T(x) = \begin{cases} \frac{1}{2} - x & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x \in (\frac{1}{2}, \infty), \end{cases}$$

respectively. Then S and T are compatible of type (B), but they are neither compatible nor compatible of type (A).

EXAMPLE 2.3. Let $X = [0, \infty)$ be a metric space with the Euclidean metric $d(x, y) = |x - y|$. Define the mappings $S, T : X \to X$ by

$$S(x) = \begin{cases} 1 + x & \text{if } x \in [0, 1] \\ 4 & \text{if } x = 1 \\ 0 & \text{if } x \in (1, \infty), \end{cases} \qquad T(x) = \begin{cases} 1 - x & \text{if } x \in [0, 1] \\ 3 & \text{if } x = 1 \\ 1 & \text{if } x \in (1, \infty), \end{cases}$$

respectively. Then S and T are compatible of type (B), but they are neither compatible nor compatible of type (A).

EXAMPLE 2.4. Let $X = [0, 1]$ be a metric space with the Euclidean metric $d(x, y) = |x - y|$. Define the mappings S and $T : X \to X$ by

$$S(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{if } x \in [\frac{1}{2}, 1], \end{cases} \qquad T(x) = \begin{cases} 1 - x & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

respectively. Then S and T are not compatible, but they are compatible of types (A) and (B).

REMARK. We note that the definitions of compatible mappings and compatible mappings of types (A) and (B) are independent for discontinuous mappings, and the mappings S and T in Examples 2.1~2.3 have not a common fixed point in X, but the mappings S and T in Example 2.4 have a common fixed point 1 in X.

III. Common fixed point theorems

Let (X, d) be a metric space. Let A, B, $S, T : X \to X$ be mappings satisfying the following conditions:
(3.1) $A(X) \subset T(X)$, $B(X) \subset S(X)$ and

$$
\begin{aligned}
d(Ax, By) \leq \max\{&d(Ax, Sx), d(By, Ty), d(Ax, Ty), \\
&\frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\} \\
&- w(\max\{d(Ax, Sx), d(By, Ty), d(Ax, Ty), \\
&\frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\})
\end{aligned} \tag{3.2}
$$

for all $x, y \in X$, where $w : R^+ \to R^+$ is a continuous function such that $0 < w(r) < r$ for all $r \in R^+ - \{0\}$.

By (3.1), since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$
\begin{cases} y_{2n} = Tx_{2n+1} = Ax_{2n} \\ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \end{cases} \tag{3.3}
$$

for $n = 0, 1, 2, \cdots$.

LEMMA 3.1. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2). *Then*

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0,$$

where $\{y_n\}$ is the sequence defined by (3.3).

Proof. Let $d_n = d(y_n, y_{n+1})$ for $n = 0, 1, 2, \cdots$. Now, we prove that the sequence $\{d_n\}$ is non-increasing in R^+, i.e., $d_n \leq d_{n-1}$ for

$n = 1, 2, \cdots$. By (3.2), we have

$$\begin{aligned}
d_{2n} &= d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1}) \\
&\le \max\{d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Ax_{2n}, Tx_{2n+1}), \\
&\quad \frac{1}{2}[d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})]\} \\
&\quad - w(\max\{d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\
&\quad d(Ax_{2n}, Tx_{2n+1}), \\
&\quad \frac{1}{2}[d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})]\}) \\
&= \max\{d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n}), \\
&\quad \frac{1}{2}d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})]\} \\
&\quad - w(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n}), \\
&\quad \frac{1}{2}[d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})]\}) \\
&= \max\{d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \\
&\quad \frac{1}{2}[d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]\} \\
&\quad - w(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \\
&\quad \frac{1}{2}[d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]\}) \\
&= \max\{d_{2n-1}, d_{2n}, \frac{1}{2}(d_{2n} + d_{2n-1})\} \\
&\quad - w(\max\{d_{2n-1}, d_{2n}, \frac{1}{2}(d_{2n} + d_{2n-1})\}).
\end{aligned}$$

If $d_{2n} > d_{2n-1}$ for any n, then we have $d_{2n} \le d_{2n} - w(d_{2n}) < d_{2n}$, which is a contradiction. Therefore, we have

$$d_{2n} \le d_{2n-1} - w(d_{2n-1}). \tag{3.5}$$

Similarly, we have

$$d_{2n+1} \le d_{2n} - w(d_{2n}). \tag{3.6}$$

From (3.5) and (3.6), it follows that, for every $n \in N$, $d_{n+1} \leq d_n - w(d_n)$, which implies that

$$\sum_{i=1}^{n} w(d_i) \leq \sum_{i=1}^{n} (d_i - d_{i+1}) = d_0 - d_{n+1} \leq d_0.$$

Therefore, the series $\sum_{i=0}^{\infty} w(d_i)$ converges and so $\lim_{n\to\infty} w(d_n) = 0$. Since $\{d_n\}$ is non-increasing in R^+, it converges to the limit p. Suppose that $p > 0$. Then, since w is continuous, $\lim_{n\to\infty} w(d_n) = w(p) = 0$, which is a contradiction and so $p = 0$. This completes the proof.

LEMMA 3.2. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2). *Then sequence $\{y_n\}$ defined by* (3.3) *is a Cauchy sequence in X.*

Proof. By virtue of Lemma 2.1, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. In order to show that $\{y_n\}$ is a Cauchy sequence in X, suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \geq 2k$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \tag{3.7}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (3.7), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon, \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{3.8}$$

Then for each even integer $2k$, we have

$$\begin{aligned} \epsilon &\leq d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) \\ &\quad + d(y_{2m(k)-1}, y_{2m(k)}). \end{aligned}$$

By Lemma 2.1 and (3.8), it follows that

$$d(y_{2n(k)}, y_{2m(k)}) \to \epsilon \quad \text{as } k \to \infty. \tag{3.9}$$

By the triangle inequality, we have

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)-1}, y_{2m(k)}),$$

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1}).$$

From Lemma 2.1 and (3.9), as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \epsilon, \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \to \epsilon. \tag{3.10}$$

Therefore, by (3.2) and (3.3), we have

$$\begin{aligned}
&d(y_{2n(k)}, y_{2m(k)}) \\
&\le d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)}) \\
&= d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1}) \\
&\le d(y_{2n(k)}, y_{2n(k)+1}) + \max\{d(Ax_{2m(k)}, Sx_{2m(k)}), \\
&\quad d(Bx_{2n(k)+1}, Tx_{2n(k)+1}), d(Ax_{2m(k)}, Tx_{2n(k)+1}), \\
&\quad \frac{1}{2}[d(Ax_{2m(k)}, Tx_{2n(k)+1}) + d(Bx_{2n(k)+1}, Sx_{2m(k)})]\} \\
&\quad - w(\max\{d(Ax_{2m(k)}, Sx_{2m(k)}), d(Bx_{2n(k)+1}, Tx_{2n(k)+1}), \\
&\quad d(Ax_{2m(k)}, Tx_{2n(k)+1}), \\
&\quad \frac{1}{2}[d(Ax_{2m(k)}, Tx_{2n(k)+1}) + d(Bx_{2n(k)+1}, Sx_{2m(k)})]\}) \\
&= d(y_{2n(k)}, y_{2n(k)+1}) + \max\{d(y_{2m(k)}, y_{2m(k)-1}), \\
&\quad d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)}, y_{2n(k)}), \\
&\quad \frac{1}{2}[d(y_{2m(k)}, y_{2n(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})]\} \\
&\quad - w(\max\{d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), \\
&\quad d(y_{2m(k)}, y_{2n(k)}), \\
&\quad \frac{1}{2}[d(y_{2m(k)}, y_{2n(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})]\}).
\end{aligned} \tag{3.11}$$

Since w is continuous, as $k \to \infty$ in (3.11), from (3.9) and (3.10), it follows that

$$\begin{aligned}
\epsilon &\le 0 + \max\{0, 0, \epsilon, \frac{1}{2}(\epsilon + \epsilon)\} - w(\max\{0, 0, \epsilon, \frac{1}{2}(\epsilon + \epsilon)\}) \\
&= \epsilon - w(\epsilon) < \epsilon.
\end{aligned}$$

This means that $w(\epsilon) \leq 0$, which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X and so $\{y_n\}$ is also a Cauchy sequence in X. This completes the proof.

THEOREM 3.3. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying* (3.1), (3.2) *and the following conditions* (3.12) *and* (3.13):
(3.12) *one of A, B, S and T is continuous,*
(3.13) *the pairs $\{A, S\}$ and $\{B, T\}$ are compatible of type (B).*
Then A, B, S and T have a unique common fixed point z in X.

Proof. By Lemma 3.2, the sequence $\{y_n\}$ defined by (3) is a Cauchy sequence in X and so, since (X, d) is complete, it converges to a point z in X. The subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the point z.

Now, suppose that T is continuous. Since B and T are compatible of type (B), by Proposition 2.10, as $n \to \infty$,

$$BBx_{2n+1}, \quad TTx_{2n+1}, \quad TBx_{2n+1} \to Tz.$$

Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (3.2), we have

$$\begin{aligned} &d(Ax_{2n}, BBx_{2n+1}) \\ &\leq \max\{d(Ax_{2n}, Sx_{2n}), d(BBx_{2n+1}, TBx_{2n+1}), \\ &\quad d(Ax_{2n}, TBx_{2n+1}), \\ &\quad \frac{1}{2}[d(Ax_{2n}, TBx_{2n+1}) + d(BBx_{2n+1}, Sx_{2n})]\} \\ &\quad - w(\max\{d(Ax_{2n}, Sx_{2n}), d(BBx_{2n+1}, TBx_{2n+1}), \\ &\quad d(Ax_{2n}, TBx_{2n+1}), \\ &\quad \frac{1}{2}[d(Ax_{2n}, TBx_{2n+1}) + d(BBx_{2n+1}, Sx_{2n})]\}. \end{aligned} \tag{3.14}$$

Taking $n \to \infty$ in (3.14), if $Tz \neq z$, then we have

$$\begin{aligned} d(z, Tz) \leq \max\{&d(z, z), d(Tz, Tz), d(z, Tz), \\ &\frac{1}{2}[d(z, Tz) + d(Tz, z)]\} \\ &- w(\max\{d(z, z), d(Tz, Tz), d(z, Tz), \\ &\frac{1}{2}[d(Tz, Tz) + d(Tz, z)]\} \end{aligned}$$

or

$$d(Tz,z) \le d(Tz,z) - w(d(Tz,z)) < d(Tz,z),$$

which is a contradiction. Thus we have $Tz = z$. Again replacing x by x_{2n} and y by z in (3.2), we have

$$\begin{aligned} &d(Ax_{2n}, Bz) \\ &\le \max\{d(Ax_{2n}, Sx_{2n}), d(Bz,Tz), d(Ax_{2n},Tz), \\ &\quad \frac{1}{2}[d(Ax_{2n},Tz) + d(Bz, Sx_{2n})]\} \\ &\quad - w(\max\{d(Ax_{2n}, Sx_{2n}), d(Bz,Tz), d(Ax_{2n},Tz), \\ &\quad \frac{1}{2}[d(Ax_{2n},Tz) + d(Bz, Sx_{2n})]\}). \end{aligned} \tag{3.15}$$

Taking $n \to \infty$ in (3.15), if $Bz \neq z$, then we have

$$\begin{aligned} d(z,Bz) &\le \max\{0, d(Bz,z), 0, \frac{1}{2}d(Bz,z)\} \\ &\qquad - w(\max\{0, d(Bz,z), 0, \frac{1}{2}d(Bz,z)\}) \end{aligned}$$

or

$$d(Bz,z) \le d(Bz,z) - w(d(Bz,z)) < d(Bz,z),$$

which means that $Bz = z$. Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $Bz = Su = z$. By using (3.2) again, we have

$$\begin{aligned} d(Au,z) = d(Au,Bz) &\le \max\{d(Au,Su), d(Bz,Tz), d(Au,Tz), \\ &\quad \frac{1}{2}(d(Au,Tz) + d(Bz,Su))\} \\ &\quad - w(\max\{d(Au,Su), d(Bz,Tz), d(Au,Tz), \\ &\quad \frac{1}{2}(d(Au,Tz) + d(Bz,Su))\}) \\ &= \max\{d(Au,z), 0, d(Au,z), \frac{1}{2}d(Au,z)\} \\ &\quad - w(\max\{d(Au,z), 0, d(Au,z), \frac{1}{2}d(Au,z)\}) \end{aligned}$$

or

$$d(Au,z) \le d(Au,z) - w(d(Au,z)) < d(Au,z),$$

which is a contradiction and so $Au = z$. But since A and S are compatible of type (B) and $Au = Su = z$, by Proposition 2.10, we have $Sz = SAu = ASu = SSu = AAu = Az$. By using (3.2), we have

$$\begin{aligned}
d(Az,z) &= d(Az,Bz) \\
&\le \max\{d(Az,Sz), d(Bz,Tz), d(Az,Tz), \\
&\quad \frac{1}{2}[d(Az,Tz) + d(Bz,Sz)]\} \\
&\quad w(\max\{d(Az,Sz), d(Bz,Tz), d(Az,Tz), \\
&\quad \frac{1}{2}d(Az,Tz) + d(Bz,Sz)]\}) \\
&= \max\{0, 0, d(Az,z), \frac{1}{2}[d(Az,z) + d(Az,z)]\} \\
&\quad - w(\max\{0, 0, d(Az,z), \frac{1}{2}[d(Az,z) + d(Az,z)]\})
\end{aligned} \tag{3.16}$$

or

$$d(Az,z) = d(Az,z) - w(d(Az,z)) < d(Az,z),$$

which is a contradiction and so $Az = z$. Therefore, $Az = Bz = Sz = Tz = z$, that is, z is a common fixed point of A, B, S and T. The uniqueness of the common fixed point z follows easily from (3.2). Similarly, we can also complete the proof when A or B or T is continuous. This completes the proof.

If we put $A = B$ in Theorem 3.3, we have the following:

THEOREM 3.4. *Let S and T be continuous mappings of a complete metric space (X,d) into itself. Then S and T have a common fixed point in X if and only if there exist a continuous mapping $A : X \to S(X) \cap T(X)$ such that*

(3.17) *the pairs $\{A,S\}$ and $\{A,T\}$ are compatible of type (B),*

$$\begin{aligned}
d(Ax,Ay) &\le \max\{d(Ax,Sy), d(Ay,Ty), d(Ax,Ty), \\
&\quad \frac{1}{2}[d(Ax,Ty) + d(Ay,Sx)]\} \\
&\quad - w(\max\{d(Ax,Sy), d(Ay,Ty), d(Ax,Ty), \\
&\quad \frac{1}{2}[d(Ax,Ty) + d(Ay,Sx)]\})
\end{aligned} \tag{3.18}$$

for all $x, y \in X$. Indeed, A, S and T have a unique common fixed point in X.

Proof. Since

$$A(X) \subset S(X) \cap T(X) \subset S(X)$$
$$A(X) \subset S(X) \cap T(X) \subset T(X),$$

from Theorem 3.3, it follows that S and T have a common fixed point in X.

Conversely, let $z \in X$ be a fixed point of S and T, i.e., $Sz = Tz = z$ and define $Ax = z$ for all $x \in X$. Then A is a continuous function from X into $S(X) \cap T(X)$. Moreover, we have, for all $x \in X$,

$$ASx = z, \quad SAx = Sz = z,$$
$$ATx = z, \quad TAx = Tz = z$$

and so $AS = SA$ and $AT = TA$, i.e., the pairs A, S and A, T are commuting. Therefore, the pairs A, S and B, T are compatible of type (A). On the other hand, the condition (3.19) holds also. This completes the proof.

If we put $A = B$ and $S = T$ in Theorem 3.3, we have the following:

COROLLARY 3.5. *Let S be a continuous mapping from a complete metric space (X, d) into itself. Then S has a fixed point in X if and only if there exists a continuous mapping $A : X \to S(X)$ such that*
(3.19) *the pair $\{A, S\}$ is compatible of type (B),*

$$\begin{aligned} d(Ax, Ay) \le \max\{&d(Ax, Sy), d(Ay, Sy), d(Ax, Sy), \\ &\frac{1}{2}[d(Ax, Sy) + d(Ay, Sx)]\} \\ &- w(\max\{d(Ax, Sy), d(Ay, Sy), d(Ax, Sy), \\ &\frac{1}{2}[d(Ax, Sy) + d(Ay, Sx)]\} \end{aligned} \tag{3.20}$$

for all $x, y \in X$, where $w : R^+ \to R^+$ is a continuous function such that $0 < w(r) < r$ for all $r \in R^+ - \{0\}$. Indeed, S and T have a unique common fixed point in X.

Putting $A = I_X$ (:the identity mapping on X) in Theorem 3.4' and $S = T$ in Corollary 3.5, respectively, we have the following:

COROLLARY 3.6. *Let S and T be continuous mappings of a complete metric space (X, d) into itself. Then S and T have a common fixed point in X if and only if*

$$(3.21)\qquad \begin{aligned} d(x,y) \leq \max\{&d(x,Sx), d(y,Ty), d(Sx,Ty), \\ &\frac{1}{2}[d(x,Ty) + d(y,Sx)]\} \\ &w(\max\{d(x,Sx), d(y,Ty), d(Sx,Ty), \\ &\frac{1}{2}[d(x,Ty) + d(y,Sx)]\}) \end{aligned}$$

for all $x, y \in X$, where $w : R^+ \to R^+$ is a continuous function such that $0 < w(r) < r$ for all $r \in R^+ - \{0\}$. Indeed, S and T has a unique common fixed point in X.

COROLLARY 3.7. *Let S be a continuous mappings of a complete metric space (X, d) into itself. Then S has a fixed point in X if and only if*

$$(3.22)\qquad \begin{aligned} d(x,y) \leq \max\{&d(x,Sx), d(y,Sy), d(x,y), \\ &\frac{1}{2}[d(x,Sy) + d(y,Sx)]\} \\ &- w(\max\{d(x,Sx), d(y,Sy), d(x,y), \\ &\frac{1}{2}[d(x,Sy) + d(y,Sx)]\}) \end{aligned}$$

for all $x, y \in X$, where $w : R^+ \to R^+$ is a continuous function such that $0 < w(r) < r$ for all $r \in R^+ - \{0\}$.

IV. Convergence of self-mappings and fixed points

In this section, by using Theorem 3.3, we give some convergence theorems for sequences of mappings from a metric space (X, d) into itself satisfying some condition.

THEOREM 4.1. *Let $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequence of mappings from a metric space (X, d) into itself such that $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to self-mappings A, B, S and T on*

X, respectively. Suppose that, for $n = 1, 2, \cdots$, z_n is a unique common fixed point of A_n, B_n, S_n and T_n and the self-mappings A, B, S and T satisfy the following condition:

$$
\begin{aligned}
d(Ax, By) \leq \max\{&d(Ax, Sy), d(By, Ty), d(Ax, Ty), \\
&\frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\} \\
&- w(\max\{d(Ax, Sy), d(By, Ty), d(Ax, Ty), \\
&\frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\})
\end{aligned}
\tag{4.1}
$$

for all $x, y \in X$, where $w : R^+ \to R^+$ is a continuous function such that $w(r) = \alpha r$ for all $r \in R^+ - \{0\}$ and $\alpha \in (0, 1)$. If z is a common fixed point of A, B, S and T and $\sup\{d(z_n, z)\} < +\infty$, then $z_n \to z$ as $n \to \infty$.

Proof. Let $\epsilon_i > 0$ for $i = 1, 2$. Since $\{A_n\}$ and $\{S_n\}$ converge uniformly to self-mappings A and S on X, respectively there exist positive integers N_1, N_2 such that, for all $x \in X$,

$$
\begin{aligned}
d(A_n x, Ax) &< \epsilon_1 \quad \text{for } n \geq N_1, \\
d(S_n x, Sx) &< \epsilon_2 \quad \text{for } n \geq N_2,
\end{aligned}
$$

respectively. Choose $N = \max\{N_1, N_2\}$ and $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. For $n \geq N$, we have

$$
\begin{aligned}
&d(z_n, z) \\
&= d(A_n z_n, Bz) \leq d(A_n z_n, Az_n) + d(Az_n, Bz) \\
&\leq d(A_n z_n, Az_n) + \max\{d(Az_n, Sz_n), \\
&\quad d(Bz, Tz), d(Az_n, Tz), \ \frac{1}{2}[A(Az_n, Tz) + d(Bz, Sz_n)]\} \\
&\quad - w(\max\{d(Az_n, Sz_n), d(Bz, Tz), d(Az_n, Tz), \\
&\quad \frac{1}{2}[d(Az_n, Tz) + d(Bz, Sz_n)]\}) \\
&\leq d(A_n z_n, Az_n) + \max\{d(Az_n, A_n z_n) + d(A_n z_n, Sz_n), 0, \\
&\quad d(Az_n, A_n z_n) + d(A_n z_n, Tz), \\
&\quad \frac{1}{2}[d(Az_n, S_n z_n) + d(S_n z_n, Tz)
\end{aligned}
$$

$$
\begin{aligned}
&+ d(Bz, S_n z_n) + d(S_n z_n, S z_n)]\} \\
&- w(\max\{d(Az_n, A_n z_n) + d(A_n z_n, S z_n), 0, \\
&d(Az_n, A_n z_n) + d(A_n z_n, Tz), \\
&\frac{1}{2}[d(Az_n, S_n z_n) + d(S_n z_n, Tz) \\
&+ d(Bz, S_n z_n) + d(S_n z_n, S z_n)]\}) \\
= {}& d(A_n z_n, Az_n) + \max\{d(A_n z_n, Az_n) + d(S_n z_n, S z_n), 0, \\
&d(A_n z_n, Az_n) + d(z_n, z), \frac{1}{2}[d(A_n z_n, Az_n) + d(z_n, z) \\
&+ d(Sz_n, S_n z_n) + d(z_n, z)]\} \\
&- w(\max\{d(A_n z_n, Az_n) + d(S_n z_n, S z_n), 0, \\
&d(A_n z_n, Az_n) + d(z_n, z), \frac{1}{2}[d(A_n z_n, Az_n) + d(z_n, z) \\
&+ d(Sz_n, S_n z_n) + d(z_n, z)]\}) \\
< {}& \epsilon + \max\{2\epsilon, 0, \epsilon + d(z_n, z), \epsilon + d(z_n, z)\} \\
&- w(\max\{2\epsilon, 0, \epsilon + d(z_n, z), \epsilon + d(z_n, z)\}).
\end{aligned} \tag{4.2}
$$

From (4.2), if $d(z_n, z) > \epsilon$, then we have

$$
\begin{aligned}
d(z_n, z) &< \epsilon + \max\{2\epsilon, 0, \epsilon + d(z_n, z), \epsilon + d(z_n, z)\} \\
&\quad - w(\max\{2\epsilon, 0, \epsilon + d(z_n, z), \epsilon + d(z_n, z)\}) \\
&= \epsilon + (\epsilon + d(z_n, z) - w(\epsilon + d(z_n, z)).
\end{aligned}
$$

Since $w(t) = \alpha t$ for all $t > 0$ and $\alpha \in (0, 1)$, we have

$$w(\epsilon + d(z_n, z)) = \alpha(\epsilon + d(z_n, z)) < 2\epsilon$$

or

$$d(z_n, z) < \frac{2\epsilon - \alpha\epsilon}{\alpha}.$$

Then letting $\alpha \to 1^-$ in (4.2), it follows that $\epsilon < d(z_n, z) \le \epsilon$, which is a contradiction. Therefore for $n \ge N$, $d(z_n, z) < \epsilon$. This means that $\{z_n\}$ converges to z. This completes the proof.

Similarly, we have the following:

THEOREM 4.2. *Let* $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ *and* $\{T_n\}$ *be sequences of mappings from a metric space* (X,d) *into itself satisfying the following condition:*

$$\begin{aligned} d(A_n x, B_n y) \le \max\{&d(A_n x, B_n x), d(B_n y, Ty), \\ &d(A_n x, Ty), \tfrac{1}{2}[d(A_n x, Ty) + d(B_n y, Sx)]\} \\ &- w(\max\{d(A_n x, B_n x), d(B_n y, Ty), \\ &d(A_n x, Ty), \tfrac{1}{2}[d(A_n x, Ty) + d(B_n y, Sx)]\}) \end{aligned} \tag{4.3}$$

for all $x, y \in X$, *where* $w : R^+ \to R^+$ *is a continuous function such that* $w(r) = \alpha r$ *for all* $r \in R^+ - \{0\}$ *and* $\alpha \in (0,1)$. *If* $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ *and* $\{T_n\}$ *converge uniformly to self-mappings* A, B, S *and* T *on* X, *respectively, then* A, B, S *and* T *satisfy the condition* (4.1). *Further, the sequence* $\{z_n\}$ *of unique common fixed point* z_n *of* A_n, B_n, S_n *and* T_n *converges to a unique common fixed point* z *of* A, B, S *and* T *if* $\sup\{d(z_n, z)\} < +\infty$.

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