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# ON THE EULER'S CONSTANT

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## 1. Introduction

The discovery that $1+\frac{1}{2}+\frac{1}{3}+\cdots$ is divergent, is attributed by James Bernoulli to his brother (Ars Conjectandi, p. 250, [7]) but the connection between $1+\frac{1}{2}+\cdots+\frac{1}{x}$ and $\log x$ was first established by Euler [6]. The *Euler's constant* is defined as following

$$\gamma = \lim_{n\to\infty}\left[1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right].$$

Euler gave the formula

$$1+\frac{1}{2}+\cdots+\frac{1}{x} = \gamma+\log x+\frac{1}{2x}-\frac{B_1}{2x^2}+\frac{B_2}{4x^4}-\frac{B_3}{6x^6}+\cdots$$

$B_1,\ B_2,\ldots$ being Bernoulli numbers, from by putting $x=10$ he calculated $\gamma = 0.57721566\ 49015325\ldots$ [11]. The value of Euler's constant was given by Mascheroni in 1790 with 32 figures as follows:

$$\gamma = 0.57721\ 56649\ 01532\ 86061\ 81120\ 90082\ 39\ldots.$$

In 1809, Soldner computed the value of $\gamma$ as

$$\gamma = 0.57721\ 56649\ 01532\ 86060\ 6065\ldots$$

which differs from Mascheroni's value in the twentieth place. In fact Mascheroni's value turned out to be not correct. However, maybe since Mascheroni's error has led to eight additional calculations of this constant, $\gamma$ is often called the *Euler-Mascheroni's constant.* Gauss in 1813 computed the 23 first decimals; in 1860 Adams [1] published the 260 first decimals. The true nature of Euler's constant (whether an

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algebraic or a transcendental number) is not known. This is a part of the famous Hilbert's sventh problem. This constant $\gamma$ is involved in lots of mathematical formulas. Mathematicians have also been interested in expressing the Euler's constant as integral forms. Sometimes they could be used to evaluate other integral values, for example, see p. 23 in [3]. In this paper, first we give some known integral representations and their slightly changed forms. Next we give integeral representations for $\gamma$ of our own.

## 2. Known integral representations for $\gamma$

In [5], we have

$$\gamma = \int_0^1 \frac{1-e^{-t}}{t} dt - \int_1^\infty \frac{e^{-t}}{t} dt, \tag{2.1}$$

$$\gamma = \int_0^\infty \left( \frac{1}{1-e^{-t}} - \frac{1}{t} \right) dt \text{ (Gauss)}, \tag{2.2}$$

$$\gamma = \int_0^\infty \left[ 1 - e^{-t} - e^{-1/t} \right] t^{-1} dt, \tag{2.3}$$

$$\gamma = \int_0^\infty \left( \frac{1}{1+t} - e^{-t} \right) \frac{dt}{t} \text{ (Dirichlet)}, \tag{2.4}$$

$$\gamma = -\int_0^\infty e^{-t} \log t dt, \tag{2.5}$$

$$\gamma = -\int_0^1 \log \left( \log \frac{1}{t} \right) dt. \tag{2.6}$$

Letting $u = e^{-t}$ in (2.2) and (2.4), we have

$$\gamma = \int_0^1 \left[ (\log t)^{-1} + (1-t)^{-1} \right] dt, \tag{2.7}$$

$$\gamma = \int_0^1 \left( 1 - \frac{1}{t - t\log t} \right) \frac{1}{\log t} dt. \tag{2.8}$$

In [3], we have

$$\gamma = \frac{1}{2} + 2\int_0^\infty \frac{t}{1+t^2} \frac{dt}{e^{2\pi t}-1}, \tag{2.9}$$

which is called *Poisson's expression* of the Euler's constant. The following six formulas for $\gamma$ are reated to circular funtions. In Titchmarsh [9], we have

$$(2.10) \qquad \gamma = -2\int_0^\infty \frac{\cos t - e^{-t^2}}{t} dt.$$

In Mikolas [4], we have

$$(2.11) \qquad \gamma = \log 2 - \pi \int_0^1 \int_0^{\frac{1}{2}} \tan\frac{\pi t}{2}\left(\frac{\sin \pi t u}{\sin \pi u} - t\right) du dt.$$

In [5], we have

$$(2.12) \qquad \gamma = -\int_0^\infty \left[\cos t - (1+t^2)^{-1}\right] t^{-1} dt.$$

In [8], we have

$$(2.13) \qquad \gamma = 1 - \int_0^\infty \left[\frac{\sin t}{t} - \frac{1}{1+t}\right]\frac{dt}{t}.$$

It is known [3] that $\lim\limits_{s\to 1}[\zeta(s)-(s-1)^{-1}] = \gamma$, where $\zeta(s)$ is the Riemann zeta function. Applying this to the formulas (12) and (13) in p. 33, [3], we have

$$(2.14) \qquad \gamma = \frac{1}{2} + 2\int_0^\infty (1+t^2)^{-1/2}\left(e^{2\pi t} - 1\right)^{-1} \sin(\tan^{-1} t) dt,$$

$$(2.15) \qquad \gamma = \log 2 - 2\int_0^\infty (1+t^2)^{-1/2}(e^{2\pi t} + 1)^{-1} \sin(\tan^{-1} t) dt,$$

which is due to Jensen.

## 3. Integral representations for $\gamma$

The Euler's constant is also related to the Gauss integer $[x]$ which denotes the greatest integer $\leq x$. In [2], we have

$$\gamma = 1 - \int_1^\infty \frac{x - [x]}{x^2} dx.$$

In [10], we have

$$\gamma = \frac{1}{2} + \int_1^\infty \frac{[x] - x + \frac{1}{2}}{x^2} dx.$$

Finally we give integral representations for $\gamma$ related to the Gauss integer. To do this, recall that the *generalized zeta* function $\zeta(s,a) = \sum_{k=0}^\infty (a+k)^{-s}$ which is analytic for $\text{Re}s > 1$ and $a > 0$. In particular, $\zeta(s,1) = \sum_{k=1}^\infty k^{-s} = \zeta(s)$ is the *Riemann zeta* function. The $nth$ *Bernoulli polynomials* $B_n(x)$ are defined as

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^\infty B_n(x) z^n, \ |z| < 2\pi.$$

The first few are:

$$B_0(x) = 1, \ B_1(x) = x - \frac{1}{2}, \ B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \ B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

Now we have the following:

THEOREM 3.1. *We have the representations, for Res $> 1$,*

$$\zeta(s,a) = \frac{1}{2a^s} + \frac{a^{1-s}}{s-1} - s\int_0^\infty \frac{B_1(x-[x])}{(x+a)^{s+1}} dx,$$

$$\zeta(s,a) = \frac{1}{2a^s} + \frac{a^{1-s}}{s-1} + \frac{s}{12a^{s+1}} - \frac{s(s+1)}{2}\int_0^\infty \frac{B_2(x-[x])}{(x+a)^{s+2}} dx,$$

$$\zeta(s,a) = \frac{1}{2a^s} + \frac{a^{1-s}}{s-1} + \frac{s}{12a^{s+1}} - \frac{s(s+1)(s+2)}{6}\int_0^\infty \frac{B_3(x-[x])}{(x+a)^{s+3}} dx.$$

*Proof.* We show this theorem for more general case. We suppose first that that $f'(x)$ is continuous for $x \geq 0$, that $f > 0$, $f' < 0$, and that $f \to 0$ when $x \to \infty$. Then

$$\int_0^x |f'(t)|dt = -\int_0^x f'(t)dt = f(0) - f(x) \to f(0)$$

when $x \to \infty$, so that

$$\int_0^\infty |f'(t)|dt = f(0) < \infty.$$

If $y = x - [x]$, so that $y = x - m + 1$ for $m - 1 \leq x < m$, then $0 \leq y < 1$ and

$$A = \int_0^\infty y f'(x)dx$$

is absolutely convergent. Also

$$\begin{aligned} a_m = f(m) - \int_{m-1}^m f(x)dx &= \int_{m-1}^m \{f(m) - f(x)\}dx \\ &= \int_{m-1}^m \{f(m) - f(x)\}\frac{dy}{dx}dx \\ &= \int_{m-1}^m y f'(x)dx, \end{aligned}$$

$$\sum_{m=1}^n f(m) - \int_0^n f(x)dx = \sum_{m=1}^n a_m = \int_0^n (x - [x])f'(x)dx.$$

Secondly, suppose that $f''(x)$ is continuous for $x \geq 0$; that $f' > 0$, $f'' < 0$; and that $f' \to 0$ when $x \to \infty$. Then

$$\int_0^x |f''(t)|dt = -\int_0^x f''(t)dt = f'(0) - f'(x) \to f'(0)$$

and $|f''|$ is integrable up to $\infty$. Thus

$$B = \frac{1}{2}\int_0^\infty (y^2 - y)f''(x)dx$$

is absolutely convergent. If

$$\begin{aligned} b_m &= \frac{1}{2}\{f(m-1)+f(m)\} - \int_{m-1}^{m} f(x)dx \\ &= f(m) - \int_{m-1}^{m} \{f(x) + \frac{1}{2}f'(x)\}dx = a_m - \frac{1}{2}\int_{m-1}^{m} f'(x)dx, \end{aligned}$$

then

$$\begin{aligned} b_m &= \int_{m-1}^{m} \left(y - \frac{1}{2}\right) f'(x)dx \\ &= \frac{1}{2}\int_{m-1}^{m} f'(x)\frac{d(y^2-y)}{dx}dx = -\frac{1}{2}\int_{m-1}^{m} (y^2-y)f''(x)dx, \end{aligned}$$

since $y^2 - y \to 0$ when $x \to m-1+0$ or $x \to m-0$. Hence, summing from $m = 1$ to $m = n$, we have

$$\text{(3.1)} \quad \sum_{k=0}^{n} f(k) = \frac{1}{2}f(0) + \frac{1}{2}f(n) + \int_0^n f(x)dx + \int_0^n \left(y - \frac{1}{2}\right) f'(x)dx,$$

$$\begin{aligned} \sum_{k=0}^{n} f(k) =& \frac{1}{2}f(0) + \frac{1}{2}f(n) + \int_0^n f(x)dx + \frac{1}{12}\int_0^n f''(x)dx \\ &- \frac{1}{2}\int_0^n (y^2 - y + \frac{1}{6})f''(x)dx. \end{aligned}$$

Finally, suppose that $f^{(3)}(x)$ is continuous for $x \geq 0$; that $f'' > 0$, $f^{(3)} < 0$; and that $f'' \to 0$ when $x \to \infty$. Then

$$\int_0^x |f^{(3)}(t)|dt = -\int_0^x f^{(3)}(t)dt = f''(0) - f''(x) \to f''(0)$$

and $|f^{(3)}|$ is integrable up to $\infty$. Thus

$$\begin{aligned} C &= \frac{1}{6}\int_0^\infty \left(y^3 - \frac{3}{2}y^2 + \frac{1}{2}y\right) f^{(3)}(x)dx \\ &= \frac{1}{6}\int_0^\infty B_3(x - [x])f^{(3)}(x)dx \end{aligned}$$

is absolutely convergent. If

$$
\begin{aligned}
c_m &= \frac{1}{2}\{f(m-1)+f(m)\} - \int_{m-1}^{m} f(x)dx - \frac{1}{12}\int_{m-1}^{m} f''(x)dx \\
&= b_m - \frac{1}{12}\int_{m-1}^{m} f''(x)dx \\
&= -\frac{1}{2}\int_{m-1}^{m}\left(y^2 - y + \frac{1}{6}\right) f''(x)dx \\
&= -\frac{1}{6}\int_{m-1}^{m} f''(x)\frac{d(y^2-y+1/6)}{dx}dx \\
&= \frac{1}{6}\int_{m-1}^{m}\left(y^3 - \frac{3}{2}y^2 + \frac{1}{2}y\right) f^{(3)}(x)dx \\
&= \frac{1}{6}\int_{m-1}^{m} B_3(x-[x])f^{(3)}(x)dx,
\end{aligned}
$$

since $y^3 - \frac{3}{2}y^2 + \frac{1}{2}y \to 0$ when $x \to m-1+0$ or $x \to m-0$. Hence, summing from $m=1$ to $m=n$,

$$
\begin{aligned}
&\frac{1}{2}f(0) + f(1) + \cdots + f(n-1) \\
&\quad + \frac{1}{2}f(n) - \int_0^n f(x)dx - \frac{1}{12}\int_0^n f''(x)dx \\
&= \frac{1}{6}\int_0^n B_3(x-[x])f^{(3)}(x)dx,
\end{aligned}
$$

or we have

$$
\begin{aligned}
\sum_{k=0}^{n} f(k) &= \frac{1}{2}f(0) + \frac{1}{2}f(n) + \int_0^n f(x)dx \\
&\quad + \frac{1}{12}\int_0^n f''(x)dx + \frac{1}{6}\int_0^n B_3(x-[x])f^{(3)}(x)dx.
\end{aligned} \tag{3.2}
$$

When $n \to \infty$, it follows from (3.1) and (3.2) that

$$
\begin{aligned}
(3.3) \qquad \sum_{k=0}^{\infty} f(k) &= \frac{1}{2} f(0) + \frac{1}{2} f(\infty) + \int_0^{\infty} f(x)dx \\
&\quad + \int_0^{\infty} B_1(x-[x]) f'(x)dx, \\
\sum_{k=0}^{\infty} f(k) &= \frac{1}{2} f(0) + \frac{1}{2} f(\infty) \\
&\quad + \int_0^{\infty} f(x)dx + \frac{1}{12} \int_0^{\infty} f''(x)dx \\
&\quad - \frac{1}{2} \int_0^{\infty} B_2(x-[x]) f''(x)dx, \\
\sum_{k=0}^{\infty} f(k) &= \frac{1}{2} f(0) + \frac{1}{2} f(\infty) \\
&\quad + \int_0^{\infty} f(x)dx + \frac{1}{12} \int_0^{\infty} f''(x)dx \\
&\quad + \frac{1}{6} \int_0^{\infty} B_3(x-[x]) f^{(3)}(x)dx,
\end{aligned}
$$

if the sums and integrals in (3.3) exist.

Let $f(x) = (x+a)^{-s}$ for $s > 1$. Then this function satisfies the conditions of (3.3). So the results follow on applying this function in (3.3).

Finally we have integral representations for $\gamma$ involved in $[x]$.

THEOREM 3.2. *We have representations*

$$
\begin{aligned}
\gamma &= \frac{1}{2} - \int_0^{\infty} \frac{B_1(x-[x])}{(x+1)^2} dx, \\
\gamma &= \frac{7}{12} - \int_0^{\infty} \frac{B_2(x-[x])}{(x+1)^3} dx, \\
\gamma &= \frac{7}{12} - \int_0^{\infty} \frac{B_3(x-[x])}{(x+1)^4} dx.
\end{aligned}
$$

*Proof.* Note that $\zeta(s,1) = \zeta(s)$ and $\gamma = \lim_{s\to 1}[\zeta(s) - (s-1)^{-1}]$. Letting $a = 1$ in the formulas of Theorem 3.1, we have the desired representations for $\gamma$.

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