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EIGENVALUE PROBLEM FOR PIECEWISE HERMITE QUADRATIC SPLINE COLLOCATION

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1. Introduction

The Hermite polynomials approximation for a given function f agrees with f at some finite number of points in the domain and their first derivatives also agree with those of f.

Let N be a positive integer and let $\{x^{(k)}\}_{k=0}^{N}$ be a uniform partition of the interval $[a,b]$, that is, $x^{(k)} = a + kh, \quad k = 0,1,\cdots,N$, where the stepsize $h=(b-a)/N$. Let $\mathcal{M}_2$ be the space of piecewise Hermite quadratics on $[a,b]$ defined by

$$\mathcal{M}_2 = \{v \in C^1[a,b] \ : \ v|_{[x^{(k)},x^{(k+1)}]} \in P_2, \quad k=0,1,\cdots,N-1\},$$

and let $\mathcal{M}_2^0 = \{v \in \mathcal{M}_2 \ : \ v(a)=v(b)=0\}$, where P_2 denotes the set of all polynomials of degree ≤ 2. Note that the dimension of $\mathcal{M}_2^0$ is N.

Let the Gaussian point $\{\xi^{(k)}\}_{k=0}^{N-1}$ on $[a,b]$ be defined by

$$\xi^{(k)} = \frac{h}{2} + x^{(k)} \qquad k=0,1,\cdots,N-1.$$

Note that the linear transformation $t = \frac{1}{b-a}(x-a)$ translates the interval $[a,b]$ into $[0,1]$ (See [2, 3, 5]).
Consider the following eigenvalue problem.

$$\text{(1.1a)} \qquad -U''(\xi^{(k)}) = \lambda U(\xi^{(k)}),\ k=0,1,\cdots,N-1,\ U \in \mathcal{M}_2.$$

$$\text{(1.1b)} \qquad U(0)=U(1)=0.$$

In this paper, following the ideas appeared in [1, 4], we will show the following theorems.

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THEOREM 1. *The eigenvalue problem* (1.1) *has* N *distinct positive eigenvalues given by*

$$\lambda_0 = \frac{8}{h^2}, \tag{1.2a}$$

and

$$\lambda_j = \frac{8-8\eta_j}{3+\eta_j}h^{-2}, \quad j = 1,2,\cdots,N-1, \tag{1.2b}$$

where $\eta_j = \cos\frac{j\pi}{N}$.

THEOREM 2. *For* $x \in [x^{(k)}, x^{(k+1)}]$, $k = 0,1,\cdots,N-1$, *the corresponding eigenfunctions are given by*

$$U_0(x) = C_0(4\rho_k^2(x) - 1), \tag{1.3a}$$

and

$$\begin{aligned} U_j(x) = C_k \Big[&\{(3+\eta_j)\sqrt{1-\eta_j} + 4(1+\eta_j)\sqrt{1-\eta_j}\rho_k(x) \\ &- 4(1-\eta_j)\sqrt{1-\eta_j}\rho_k^2(x)\}\cos\frac{kj\pi}{N} \\ &+ \{(3+\eta_j)\sqrt{1+\eta_j} + 4(1-\eta_j)\sqrt{1+\eta_j}\rho_k(x) \\ &-4(1-\eta_j)\sqrt{1+\eta_j}\rho_k^2(x)\}\sin\frac{kj\pi}{N}\Big], \end{aligned} \tag{1.3b}$$

where $j = 1,2,\cdots,N-1$, $\rho_k(x) = \frac{(x-\xi^{(k)})}{h}$, $C_k = \frac{-h^2\gamma_0}{8(1-\eta_j)\sqrt{1-\eta_j}}$ *and* $C_0 = \frac{h^2}{8}\gamma_k$.

2. Proofs of main results

Let λ and U denote a real nonzero eigenvalue and the corresponding eigenfunction of (1.1), respectively, and for $x \in [x^{(k)}, x^{(k+1)}]$, $k = 0,1,\cdots,N-1$. Let

$$U_k(x) := \alpha_k + \beta_k(x-\xi^{(k)}) + \gamma_k\frac{(x-\xi^{(k)})^2}{2}. \tag{2.1a}$$

Then we have

(2.1b) $$U_k'(x) = \beta_k + \gamma_k(x - \xi^{(k)})$$

and

(2.1c) $$U_k''(x) = \gamma_k.$$

From (1.1a) with (2.1a) and (2.1c), we have

(2.2) $$-\gamma_k = \lambda\alpha_k, \quad \text{for} \quad k = 0, 1, \cdots, N-1.$$

Note that

(2.3a) $$U_{k+1}(x) = \alpha_{k+1} + \beta_{k+1}(x - \xi^{(k+1)}) + \gamma_{k+1}\frac{(x - \xi^{(k+1)})^2}{2}$$

on $[x^{(k+1)}, x^{(k+2)}]$. Then

(2.3b) $$U_{k+1}'(x) = \beta_{k+1} + \gamma_{k+1}(x - \xi^{(k+1)})$$

and

(2.3c) $$U_{k+1}''(x) = \gamma_{k+1}.$$

We will use the C^1 -condition at $x^{(k+1)}$ to get some informations about α_k, β_k and γ_k, $k = 0, 1, \cdots, N-2$.
By the continuity of U at $x^{(k+1)}$, we get

$$U_k(x^{(k+1)}) = U_{k+1}(x^{(k+1)}),$$

$$\text{i.e.,} \quad \alpha_k + \beta_k\frac{h}{2} + \gamma_k\frac{h^2}{8} = \alpha_{k+1} - \beta_{k+1}\frac{h}{2} + \gamma_{k+1}\frac{h^2}{8}.$$

With (2.2), this implies that

(2.4) $$\left(\frac{h^2}{8} - \frac{1}{\lambda}\right)\gamma_k + \beta_k\frac{h}{2} = \left(\frac{h^2}{8} - \frac{1}{\lambda}\right)\gamma_{k+1} - \beta_{k+1}\frac{h}{2}.$$

Moreover, by the continuity of U' at $x^{(k+1)}$, we have

$$U_k'(x^{(k+1)}) = U_{k+1}'(x^{(k+1)}),$$

(2.5) $$\text{i.e.,} \quad \beta_k + \gamma_k\frac{h}{2} = \beta_{k+1} - \gamma_{k+1}\frac{h}{2}.$$

Thus we can express (2.4) and (2.5) as follows.

(2.6) $$\begin{pmatrix} -r & s \\ 1 & -r \end{pmatrix}\begin{pmatrix} \beta_{k+1} \\ \gamma_{k+1} \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & r \end{pmatrix}\begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix}$$

for $k = 0, 1, \cdots, N-2$, where $r = \frac{h}{2}$, $s = \frac{h^2}{8} - \frac{1}{\lambda}$.

LEMMA 1. *The coefficients α_k, β_k and γ_k of (2.1a) satisfy the following relations.*

(a) $\alpha_k = -\frac{1}{\lambda}\gamma_k$.

(b) $r\beta_0 = s\gamma_0$.

(c) $s\gamma_{N-1} = -r\beta_{N-1}$.

(d) $\begin{pmatrix}\beta_{k+1}\\ \gamma_{k+1}\end{pmatrix} = \left(\frac{1}{s-r^2}\right)^{k+1}\begin{pmatrix} r^2+s & 2sr \\ 2r & r^2+s\end{pmatrix}^{k+1}\begin{pmatrix}\beta_0\\ \gamma_0\end{pmatrix}$,
$k = 0, 1, \cdots, N-2$.

Proof. (a): First note that (a) follows from (2.2).
(b): Since $U(0) = 0$ ($k = 0$, $x = x^{(0)}$), we have $U(0) = \alpha_0 - \beta_0\frac{h}{2} + \gamma_0\frac{h^2}{8} = 0$ From (a), $\frac{1}{\lambda}\gamma_0 - \beta_0\frac{h}{2} + \gamma_0\frac{h^2}{8} = 0$, i.e., $\left(\frac{h^2}{8} - \frac{1}{\lambda}\right)\gamma_0 = \frac{h}{2}\beta_0$. Therefore, we have (b).
(c): Since $U(1) = 0$ $(k = N-1, x = x^{(N)})$, we have

$$U(1) = \alpha_{N-1} + \beta_{N-1}\frac{h}{2} + \gamma_{N-1}\frac{h^2}{8} = 0.$$

From $\alpha_{N-1} = -\frac{1}{\lambda}\gamma_{N-1}$, we have

$$-\frac{1}{\lambda}\gamma_{N-1} + \beta_{N-1}\frac{h}{2} + \gamma_{N-1}\frac{h^2}{8} = 0, \text{ i.e., } \left(\frac{h^2}{8} - \frac{1}{\lambda}\right)\gamma_{N-1} = -\beta_{N-1}\frac{h}{2}.$$

Hence, we have (c).
(d): From (2.6), we have $\begin{pmatrix}\beta_{k+1}\\ \gamma_{k+1}\end{pmatrix} = \begin{pmatrix} -r & s \\ 1 & -r\end{pmatrix}^{-1}\begin{pmatrix} r & s \\ 1 & r\end{pmatrix}\begin{pmatrix}\beta_k\\ \gamma_k\end{pmatrix}$.
Note that the determinant D of $\begin{pmatrix} r & s \\ 1 & r\end{pmatrix}$ is as follows.

$$D = r^2 - s = \left(\frac{h}{2}\right)^2 - \left(\frac{h^2}{8} - \frac{1}{\lambda}\right) = \frac{h^2}{8} + \frac{1}{\lambda} \neq 0.$$

So,

$$\begin{aligned}
\begin{pmatrix} \beta_{k+1} \\ \gamma_{k+1} \end{pmatrix} &= \frac{1}{r^2 - s} \begin{pmatrix} -r & -s \\ -1 & -r \end{pmatrix} \begin{pmatrix} r & s \\ 1 & r \end{pmatrix} \begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix} \\
&= \frac{1}{s - r^2} \begin{pmatrix} r^2 + s & 2sr \\ 2r & r^2 + s \end{pmatrix} \begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix} \\
&= \frac{1}{s - r^2} \begin{pmatrix} r^2 + s & 2sr \\ 2r & r^2 + s \end{pmatrix} \\
&\quad \cdot \frac{1}{s - r^2} \begin{pmatrix} r^2 + s & 2sr \\ 2r & r^2 + s \end{pmatrix} \begin{pmatrix} \beta_{k-1} \\ \gamma_{k-1} \end{pmatrix} \\
&= \cdots = (\frac{1}{s - r^2})^{k+1} \begin{pmatrix} r^2 + s & 2sr \\ 2r & r^2 \end{pmatrix}^{k+1} \begin{pmatrix} \beta_0 \\ \gamma_0 \end{pmatrix}.
\end{aligned}$$

Proof of Theorem 1. Case (i): Assume that $s < 0$.
Then it is easy to verify that

$$\begin{aligned}
&\begin{pmatrix} -\frac{1}{s} & 0 \\ 0 & \frac{1}{\sqrt{-s}} \end{pmatrix} \begin{pmatrix} r^2 + s & 2sr \\ 2r & r^2 + s \end{pmatrix} \begin{pmatrix} -s & 0 \\ 0 & \sqrt{-s} \end{pmatrix} \\
&= \begin{pmatrix} r^2 + s & -2r\sqrt{-s} \\ 2r\sqrt{-s} & r^2 + s \end{pmatrix}.
\end{aligned}$$

We multiply $\frac{1}{s-r^2}$ on both sides.

$$\begin{aligned}
&\frac{1}{s - r^2} \begin{pmatrix} -\frac{1}{s} & 0 \\ 0 & \frac{1}{\sqrt{-s}} \end{pmatrix} \begin{pmatrix} r^2 + s & 2sr \\ 2r & r^2 + s \end{pmatrix} \begin{pmatrix} -s & 0 \\ 0 & \sqrt{-s} \end{pmatrix} \\
&= \frac{1}{s - r^2} \begin{pmatrix} r^2 + s & -2r\sqrt{-s} \\ 2r\sqrt{-s} & r^2 + s \end{pmatrix}.
\end{aligned}$$

Taking k-th power, we have

$$\begin{aligned}
&\begin{pmatrix} -\frac{1}{s} & 0 \\ 0 & \frac{1}{\sqrt{-s}} \end{pmatrix} (\frac{1}{s - r^2})^k \begin{pmatrix} r^2 + s & 2sr \\ 2r & r^2 + s \end{pmatrix}^k \begin{pmatrix} -s & 0 \\ 0 & \sqrt{-s} \end{pmatrix} \\
&= (\frac{1}{s - r^2})^k \begin{pmatrix} r^2 + s & -2r\sqrt{-s} \\ 2r\sqrt{-s} & r^2 + s \end{pmatrix}^k.
\end{aligned}$$

Therefore,

$$(2.7)\quad (\frac{1}{s-r^2})^k \begin{pmatrix} r^2+s & 2sr \\ 2r & r^2+s \end{pmatrix}^k$$
$$= \begin{pmatrix} -\frac{1}{s} & 0 \\ 0 & \frac{1}{\sqrt{-s}} \end{pmatrix}^{-1} \begin{pmatrix} \frac{r^2+s}{s-r^2} & \frac{-2r\sqrt{-s}}{s-r^2} \\ \frac{2r\sqrt{-s}}{s-r^2} & \frac{r^2+s}{s-r^2} \end{pmatrix}^k \begin{pmatrix} -s & 0 \\ 0 & \sqrt{-s} \end{pmatrix}^{-1}.$$

Note that $|s+r^2| < |s-r^2|$.
Let θ be such that

$$(2.8)\qquad \cos\theta = \frac{s+r^2}{s-r^2}, \quad \theta \in (0,\pi),$$

then

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(\frac{s+r^2}{s-r^2})^2} = \frac{\sqrt{-4sr^2}}{s-r^2} = \frac{2r\sqrt{-s}}{s-r^2}.$$

It follows from Lemma 1 (d), (2.7) and (2.8) that

$$\begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix} = (ST)^k \begin{pmatrix} \beta_0 \\ \gamma_0 \end{pmatrix}$$
$$= \begin{pmatrix} -s & 0 \\ 0 & \sqrt{-s} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^k \begin{pmatrix} -\frac{1}{s} & 0 \\ 0 & \frac{1}{\sqrt{-s}} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \gamma_0 \end{pmatrix},$$

where

$$S = \frac{1}{s-r^2}, \qquad T = \begin{pmatrix} r^2+s & 2sr \\ 2r & r^2+s \end{pmatrix}.$$

Since

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix},$$

we have

$$(2.9)\qquad \begin{cases} \beta_k & = \beta_0 \cos k\theta - \gamma_0\sqrt{-s}\sin k\theta \\ \gamma_k & = \gamma_0 \cos k\theta + \frac{\beta_0}{\sqrt{-s}} \sin k\theta, \ k = 0, 1, \cdots, N-1. \end{cases}$$

From (2.9),

$$\beta_{N-1} = \beta_0 \cos(N-1)\theta - \gamma_0\sqrt{-s}\sin(N-1)\theta,$$

$$\gamma_{N-1} = \gamma_0 \cos(N-1)\theta + \frac{\beta_0}{\sqrt{-s}}\sin(N-1)\theta.$$

Then we have

$$r\beta_{N-1} = r\beta_0 \cos(N-1)\theta - r\gamma_0\sqrt{-s}\sin(N-1)\theta, \tag{2.10}$$

and

$$s\gamma_{N-1} = s\gamma_0 \cos(N-1)\theta + s\frac{\beta_0}{\sqrt{-s}}\sin(N-1)\theta. \tag{2.11}$$

(2.10) and (2.11) imply that

$$0 = 2r\beta_0 \cos(N-1)\theta - (r\gamma_0\sqrt{-s} + \beta_0\sqrt{-s})\sin(N-1)\theta, \tag{2.12}$$

since $r\beta_0 = s\gamma_0, \quad s\gamma_{(N-1)} = -r\beta_{(N-1)}$.
From Lemma 1, (2.8) and (2.12), we have

$$\begin{aligned}\cot(N-1)\theta = \frac{\cos(N-1)\theta}{\sin(N-1)\theta} &= \frac{r\gamma_0\sqrt{-s} + \beta_0\sqrt{-s}}{2r\beta_0} \\ &= \frac{\frac{s}{\beta_0}\gamma_0^2\sqrt{-s} + \beta_0\sqrt{-s}}{2s\gamma_0} = \frac{\sqrt{-s}}{2s\gamma_0}\cdot\frac{(s\gamma_0^2+\beta_0^2)}{\beta_0} \\ &= \frac{\sqrt{-s}}{2sr}(r^2+s) - \cot\theta = \cot(-\theta),\end{aligned}$$

that yields

$$N\theta = j\pi, \qquad j = 1, 2, \cdots, N-1.$$

Set

$$\eta_j = \cos\theta = \cos\frac{j\pi}{N} = \frac{s+r^2}{s-r^2} = \frac{\left(\frac{h^2}{8} - \frac{1}{\lambda}\right) + (\frac{h}{2})^2}{\left(\frac{h^2}{8} - \frac{1}{\lambda}\right) - (\frac{h}{2})^2} = \frac{\frac{3}{8}h^2 - \frac{1}{\lambda}}{-\frac{1}{8}h^2 - \frac{1}{\lambda}}.$$

With $\lambda h^2 = a$, we have $\eta_j = \frac{3\lambda h^2-8}{-\lambda h^2-8} = \frac{3a-8}{-a-8}$.
This implies that $(-a-8)\eta_j = 3a-8$, i.e. $a = \frac{8-8\eta_j}{3+\eta_j}$.
Therefore, $\lambda h^2 = \frac{8-8\eta_j}{3+\eta_j}$; hence, $\lambda_j = \frac{8-8\eta_j}{3+\eta_j}h^{-2}$. Moreover, because of $\eta_j = \cos\frac{j\pi}{N}$, $\quad -1 < \eta_j < 1$, we have $0 < \frac{\lambda_j}{8}h^2 < 1$.

Case (ii): Assume that $s = 0$.
Then we have $\frac{h^2}{8} - \frac{1}{\lambda} = 0$. Thus $\lambda_0 = \frac{8}{h^2}$.

Proof of Theorem 2. Case (i): Assume that $s < 0$.
Note that

$$s = \frac{h^2}{8} - \frac{1}{\lambda} = \frac{h^2}{8} - \frac{(3+\eta_j)h^2}{8-8\eta_j} = \frac{h^2}{8}\left(1 - \frac{3+\eta_j}{1-\eta_j}\right)$$
$$= \frac{h^2}{8}\cdot\frac{-2(1+\eta_j)}{1-\eta_j} = -\frac{h^2}{4}\cdot\frac{1+\eta_j}{1-\eta_j}.$$

Hence $\sqrt{-s} = \frac{h}{2}\sqrt{\frac{1+\eta_j}{1-\eta_j}}$.
Recall that $\alpha_k = -\frac{1}{\lambda}\gamma_k$, $r\beta_0 = s\gamma_0$, $\theta = \frac{j\pi}{N}$, $\quad j = 1, 2, \cdots, N-1$.
Then from $(2.1a)$,

$$U_k(x) = \alpha_k + \beta_k(x-\xi^{(k)}) + \gamma_k\frac{(x-\xi^{(k)})^2}{2}$$
$$= -\frac{1}{\lambda}\gamma_k + \beta_k(x-\xi^{(k)}) + \gamma_k\frac{(x-\xi^{(k)})^2}{2}$$
$$= -\frac{(3+\eta_j)h^2}{8-8\eta_j}\gamma_k + h\rho_k(x)\beta_k + \frac{h^2}{2}\rho_k(x)^2\gamma_k$$
$$= -\frac{(3+\eta_j)h^2}{8-8\eta_j}\left(\frac{\beta_0}{\sqrt{-s}}\sin k\theta + \gamma_0\cos k\theta\right) + h\rho_k(x)(\beta_0\cos k\theta$$
$$-\gamma_0\sqrt{-s}\sin k\theta) + \frac{h^2}{2}\rho_k^2(x)\left(\frac{\beta_0}{\sqrt{-s}}\sin k\theta + \gamma_0\cos k\theta\right)$$
$$= -\frac{\sqrt{-s}\gamma_0 h}{4(1-\eta_j)}(3+\eta_j+4(1-\eta_j)\rho_k(x) - 4(1-\eta_j)\rho_k^2(x))$$
$$\sin\frac{kj\pi}{N} - \frac{\gamma_0 h^2}{8(1-\eta_j)}(3+\eta_j+4(1+\eta_j)\rho_k(x) - 4(1-\eta_j)$$
$$\rho_k^2(x))\cos\frac{kj\pi}{N}$$

$$= \frac{-h^2\gamma_0}{8(1-\eta_j)\sqrt{1-\eta_j}}[\sqrt{1+\eta_j}\{3+\eta_j+4(1-\eta_j)\rho_k(x).$$
$$-4(1-\eta_j)\rho_k^2(x)\}\sin\frac{kj\pi}{N}+\sqrt{1-\eta_j}\{3+\eta_j$$
$$+4(1+\eta_j)\rho_k(x).-4(1-\eta_j)\rho_k^2(x)\}\cos\frac{kj\pi}{N}].$$

Therefore, we have

$$\begin{aligned} U_j(x) = C_k[&\{(3+\eta_j)\sqrt{1-\eta_j}+4(1+\eta_j)\sqrt{1-\eta_j}\rho_k(x). \\ &-4(1-\eta_j)\sqrt{1-\eta_j}\rho_k^2(x)\}\cos\frac{kj\pi}{N} \\ &+\{(3+\eta_j)\sqrt{1+\eta_j}+4(1-\eta_j)\sqrt{1+\eta_j}\rho_k(x) \\ &-4(1-\eta_j)\sqrt{1+\eta_j}\rho_k^2(x)\}\sin\frac{kj\pi}{N}], \end{aligned}$$

where

$$\rho_k(x) = \frac{(x-\xi^{(k)})}{h} \quad \text{and} \quad C_k = \frac{-h^2\gamma_0}{8(1-\eta_j)\sqrt{1-\eta_j}}.$$

Case (ii): Assume that $s=0$.
From (2.6),

$$\begin{pmatrix} -r & 0 \\ 1 & -r \end{pmatrix}\begin{pmatrix} \beta_{k+1} \\ \gamma_{k+1} \end{pmatrix} = \begin{pmatrix} r & 0 \\ 1 & r \end{pmatrix}\begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix}, \quad \text{for} \quad k=0,1,\cdots,N-2,$$

i.e., $-r\beta_{k+1}=r\beta_k$ and $\beta_{k+1}-r\gamma_{k+1}=\beta_k+r\gamma_k$. Hence $\beta_k=0$. So $s\gamma_0=r\beta_0=0$. Therefore $-r\gamma_{k+1}=r\gamma_k$, i.e., $\gamma_{k+1}=-\gamma_k$.

Thus

$$\begin{aligned} U(x) =& \alpha_k+\beta_k(x-\xi^{(k)})+\gamma_k\frac{(x-\xi^{(k)})^2}{2} \\ =& \alpha_k+\gamma_k\frac{(x-\xi^{(k)})^2}{2} \\ =& -\frac{h^2}{8}\gamma_k+\gamma_k\frac{(x-\xi^{(k)})^2}{2} \\ =& -\frac{h^2}{8}\gamma_k\left(1-\frac{4(x-(\xi^{(k)})^2}{h^2}\right). \end{aligned}$$

Let $\rho_k(x) = \frac{(x-\xi^{(k)})}{h}$ and $C_0 = \frac{h^2}{8}\gamma_k$. Then we have

$$U_0(x) = C_0(4\rho_k^2(x) - 1).$$

REMARK. Assume that $s > 0$. From Lemma 1, $r\beta_0 = s\gamma_0$, $s\gamma_{N-1} = -r\beta_{N-1}$ and

$$\begin{pmatrix} \beta_{k+1} \\ \gamma_{k+1} \end{pmatrix} = \frac{1}{s - r^2} \begin{pmatrix} r^2 + s & 2sr \\ 2r & r^2 + s \end{pmatrix} \begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix}.$$

Since $r > 0$ and $s > 0$ and $s - r^2 = -\left(\frac{h^2}{8} + \frac{1}{\lambda}\right) \neq 0$, β_k and γ_k must have the same sign. But $s\gamma_{N-1} = -r\beta_{N-1}$, hence β_k and γ_k have different signs.
This is a contradiction. Therefore this case cannot happen.

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