Honam Mathematical J. **19**(1997), No. 1,pp. 125–130

ON THE THEORY OF SELECTIONS

SEUNG WOO LEE

Dept. of Mathematics,
Wonkwang University, Iksan, Chonbuk 570-749, Korea.

Abstract In this paper, we give a characterization of collectionwise normality using continuous functions. More precisely, we give a new and short proof of the Dowker's theorem using selection theory that a T_1 space X is collectionwise normal iff every continuous mapping of every closed subset F of X into a Banach space can be continuously extended over X. This is also a generalization of Tietze's extension theorem.

1. Introduction

An extension problem is one of the important branch in general topology. We consider a following general extension problem in general topology:

Let X and Y be two topological spaces with $A \subseteq X$ closed, and let $f : A \to Y$ be continuous. Under what conditions on X and Y, does f have a continuous extension over X (or at least over some open $U \supseteq A$) ?

THEOREM 1.1. *(Tietze's extension theorem) Let X be a normal space. Then for every closed $A \subseteq X$ and every continuous function $f : A \to \mathbb{R}$, there exists a continuous extension of f over X.*

In this paper, we are going to generalize Tietze's extension theorem by use of a new point of view due to E. Michael[3,4,5]. He considered additional requirements on f as follows:

Received April 11, 1997.

1991 Mathematics Subject Classification : 54A25.

Key words and phrases : collectionwise normal, extension, selection.

The author's research was supported by Wonkwang University Research Grant, 1996.

For every $x \in X$, $f(x)$ must be an element of a pre-assigned subset of Y. This new problem is called the *selection problem*. We will only look at the theory of selection based on Michael's papers from the mid-50s. There are four main selection theorems: zero-dimensional, convex-valued, compact-valued, and finite-dimensional theorems. This paper is focused only on a convex-valued selection theorem.

Theorem 2.9 [3, Theorem 3.2″] says that for a T_1 space X the following are equivalent:

(a) X is paracompact

(b) If Y is a Banach space, then every lower semi-continuous carrior for X to the family of non-empty, closed, convex subsets of Y admits a selection.

This theorem characterizes paracompactness using continuous functions rather than covering properties. Since the continuous functions are relatively easier to handle covering properties, the above Michael's theorem is important and applicable to certain proofs. In this paper, we also give a characterization of collectionwise normality using continuous functions as in Michael's paper.

2. Preliminaries

Let X and Y be topological spaces, and 2^Y be the family of nonempty subsets of Y. As far as topological terminology, we follow [2].

DEFINITION 2.1. A map $\phi : X \to 2^Y$ is called a *carrior* (or set-valued mapping). A *selection* for $\phi : X \to 2^Y$ is a continuous map $f : X \to Y$ such that $f(x) \in \phi(x)$ for every $x \in X$.

A carrior $\phi : X \to 2^Y$ is called *lower semi-continuous* (l.s.c.) if for every open set V of Y, $\phi^{-1}\{V\} = \{p \in X : \phi(p) \cap V \neq \emptyset\}$ is an open set of X.

EXAMPLES 2.2.

(a) A mapping $\phi X \to Y$ may be considered as a special carrior of X into 2^Y such that each $\phi(p)$ is a one-point subset of Y. Then this carrior is l.s.c. iff the mapping ϕ is continuous because $\phi^{-1}\{V\} = \phi^{-1}(V)$ for every subset V of Y.

(b) Let $f : X \to Y$ be an open surjection and define $\phi : Y \to \mathcal{P}(X)$ by $\phi(y) = f^{-1}(y)$. Then ϕ is l.s.c..

(c) Let $F \subseteq X$ be closed and f a continuous mapping defined over F. Then the carrior defined by

$$\phi(p) = \begin{cases} f(p), & \text{if } p \in F \\ X, & \text{if } p \in X - F \end{cases}$$

is also l.s.c.. If there is a selection g for ϕ, then it is clearly a continuous extension of f over X.

EXAMLPE 2.3[3]. Let $\psi : X \to 2^Y$, $A \subseteq X$, and let $g : A \to Y$ be a selection for $\psi \mid A$. Define $\phi : X \to 2^Y$ by

$$\phi(x) = \begin{cases} \{g(x)\}, & \text{if } x \in A \\ \psi(x), & \text{if } x \in X - A \end{cases}$$

Then $f : X \to Y$ is a selection for ϕ if and only if f is a selection for ψ which extends g.

QUESTION 2.4[3]. *Under what conditions on X, $A \subseteq X$, and $\phi : X \to 2^Y$, can every selection for $\phi \mid A$ be extened to a selection for ϕ, or at least for $\phi \mid U$ for some open $U \supseteq A$?*

We introduce a characterization of a l.s.c. carrier given by E. Michael [3].

PROPOSITION 2.5. *A carrior $\phi : X \to 2^Y$ is l.s.c. if and only if for each $x \in X$, $y \in \phi(x)$, and a neighborhood V of y, there exists a neighborhood U of x such that $\phi(x') \cap V \neq \emptyset$ for every $x' \in U$.*

The following theorem is so called a selection extension property.

THEOREM 2.6[3]. *Let X and Y be topological spaces and $\mathcal{S} \subseteq 2^Y$ be a colllection containing all single-ton sets Then the following are equivalent.*

(i) If $\phi : X \to \mathcal{S}$ is a l.s.c. carrior, then for every closed $A \subseteq X$, each selection for $\phi \mid A$ can be extended to a selection for ϕ.

(ii) Every l.s.c. carrior $\phi : X \to \mathcal{S}$ has a selection.

REMARK 2.7. Theorem 2.6 reduces the problem which deals with extending a selection to the simpler problem of merely finding one.

For the rest of this paper, we introduce the following notations which are essentially given by E. Michael [3].

NOTATION 2.8. *For a Banach space Y, we define:*
$\mathcal{K}(Y) = \{S \in 2^Y : S \text{ is convex}\}$,
$\mathcal{F}(Y) = \{S \in \mathcal{K}(Y) : S \text{ is closed}\}$,
$\mathcal{C}(Y) = \{S \in \mathcal{F}(Y) : S \text{ is compact or } S = Y\}$,
$\mathcal{D}(Y) = \{S \in \mathcal{K}(Y) : S$ *is either finite-dimensional, or closed, or has an interior point*$\}$.

We note that $\mathcal{C}(Y) \subseteq \mathcal{F}(Y) \subseteq \mathcal{D}(Y) \subseteq \mathcal{K}(Y) \subseteq 2^Y$,
$\mathcal{K}(\mathbb{R}) = \mathcal{D}(\mathbb{R})$, where $\mathbb{R}$ is the real line.

THEOREM 2.9. *[3, Theorem 3.2″] Let X be a T_1-space. Then the following are equivalent.*

(a) X is T_2 paracompact

(b) If Y is a Banach space, then every l.s.c. $\phi : X \to \mathcal{F}(Y)$ has a selection.

Before turning to our next section, observe that the assumption in Theorem 2.9 "closedness" cannot simply be omitted, even if $X = [0, 1]$.

3. A Characterization of Collectionwise Normality

A topological space X is called *collectionwise normal* if X is a T_1-space and for every discrete family $\{F_\alpha : \alpha \in A\}$ of closed subsets of X there exists a discrete family $\{V_\alpha : \alpha \in A\}$ of open subsets of X such that $F_\alpha \subseteq V_\alpha$ for every $\alpha \in A$.

Clearly, every collectionwise normal space is normal. Collectionwise normality is another strengthening of normality, weaker than paracompactness.

THEOREM 3.1. *[3] For a T_1-space X, the following are equivalent.*

(a) X is collectionwise normal

(b) If Y is a Banach space, then every l.s.c. carrior $\phi : X \to \mathcal{C}(Y)$ admits a selection.

The proof of the following corollary is basically due to H. Dowker [1], but we give a new and short proof using selection theory. This is also a generalization of Tietze's extension theorem.

COROLLARY 3.2. *A T_1-space X is collectionwise normal if and only if every continuous mapping of every closed subset F of X into a Banach space can be continuously extened over X.*

Proof. ($\Rightarrow$) It is a direct consequence of the 'only if' part of Theorem 3.1.

($\Leftarrow$) To prove the 'if' part, we let $\{F_\alpha : \alpha \in A\}$ be a discrete closed collection of X. We denote by $H(A)$ the generalized Hilbert space with the index set A. Define a mapping f of the closed set $\cup\{F_\alpha : \alpha \in A\}$ of X into $H(A)$ by $f(F_\alpha) = p^\alpha$, where $p^\alpha = \langle x_{\alpha'} : \alpha' \in A\rangle, x_\alpha = 1, x_{\alpha'} = 0$ for $\alpha' \neq \alpha$. Then f is easily seen to be continuous over $\cup\{F_\alpha : \alpha \in A\}$, and hence by the hypothesis there is a continuous extension g of f over X. We consider a ball $B(p_\alpha; 1/2)$ for each p^α. Then $B(p_\alpha; 1/2) \cap B(p_{\alpha'}; 1/2) = \emptyset$ if $\alpha \neq \alpha'$. Therefore the sets $U_\alpha = g^{-1}(B(p_\alpha; 1/2)), \alpha \in A$, form a disjoint open collection in X satisfying $U_\alpha \supseteq F_\alpha$. This proves that X is collectionwise normal.

We now reprove Theorem 3.1$'$ given by E. Michael in [3] using the results in the same paper.

THEOREM 3.3. *For a T_1 space X, the following are equivalent:*

(a) X is normal

(b) Every l.s.c. carrior $\phi : X \to \mathcal{C}(\mathbb{R})$ admits a selection

(c) If Y is a separable Banach space, then every l.s.c. carrior $\phi : X \to \mathcal{C}(Y)$ admits a selection.

Proof. We will show that $(b) \Rightarrow (a) \Rightarrow (c) \Rightarrow (d)$.

$(b) \Rightarrow (a)$: Suppose the condition (b) holds. Since $\mathcal{C}(\mathbb{R}) = \mathcal{K}(\mathbb{R})$ and $\mathbb{R}$ is a Banach space, X is T_2 paracompact by Theorem 2.9, and hence X is normal.

$(a) \Rightarrow (c)$: Suppose X is normal. Let Y be a separable Banach space. We can derive condition (c) from Lemma $4.1''$ in [3] as we derived in the proof of Theorem 3.1.

$(c) \Rightarrow (b)$: It is clear because $\mathbb{R}$ is a separable Banach space.

REMARK. $(b) \Rightarrow (a)$ also follows from Tietze's extension theorem, i.e., it follows from Theorem 3.1 and Corollary 1.5 in [3].

References

[1]. C.H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. **69** (1947), 200–242.

[2]. R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[3]. E. Michael, *Continuous selections I*, Ann. of Math. **63** (1956), 361–382.

[4]. ______, *Continuous selections II*, Ann. of Math. **64** (1956), 562–580.

[5]. ______, *A theorem on semi-continuous set-valued functions*, Duke Math. J. **Vol.26, No.4** (1959), 647–656.

[6]. J. van Mill, *Infinite Dimensional Topology*, North-Holland, Amsterdam, 1989.