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ASYMPTOTIC BEHAVIOR OF BLOW–UP SOLUTION OF A LOCALIZED SEMILINEAR PARABOLIC EQUATION

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1. Introduction

The purpose of this paper is to investigate the growth rate of blow-up solution to the semilinear parabolic equation

$$\text{(P)} \qquad \begin{aligned} & u_t = u_{xx} + e^{u(0,t)}, \quad (x,t) \in (-a,a) \times (0,T) \\ & u(\pm a, t) = 0, \quad t > 0 \\ & u(x,0) = u_0(x), \quad x \in (-a,a) \end{aligned}$$

in a neighborhood of a blow-up point as t approaches the finite blow-up time $T < \infty$, where a is a positive constant.

Assume that the initial condition satisfies the followings:
(Assumption A) : $u_0(x) \in C^2[-a,a]$ is nonnegative, bounded, symmetric and $u_0(x)$ is nondecreasing in $(-a,0)$ and
(Assumption B) : $u_0''(x) + e^{u_0(0)} \geq 0,\ x \in (-a,a)$.

Then a unique solution $u(x,t)$ of (P) exists for $t < \sigma_0$ sufficiently small.

By the maximum principle, $u(x,t) \geq 0$, $(x,t) \in (-a,a) \times (0,T)$ and

$$U(t) = \max_{x \in [-a,a]} u(x,t) \tag{1}$$

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is monotone increasing in t.

If a solution $u(x,t)$ exists for all $t < \sigma(< \infty)$ and $U(\sigma^-) < \infty$, then the solution $u(x,t)$ can be uniquely continued into some interval $0 < t < \sigma + \epsilon$ with $\epsilon > 0$ sufficiently small.

Let T be the supremum of all σ such that the solution exists for all $t < \sigma$. We assume $T < \infty$. Then

$$U(T^-) = \infty.$$

In this case, T is called *the finite blow-up time.*

The problem (P) aries in a model for ignition(see Bebernes and Kassoy[1]). Wang and Chen[7] recently characterized the growth rate of asymptotic behavior for blow-up solution of

$$(2) \qquad \begin{aligned} &u_t = u_{xx} + u^p(0,t), \quad (x,t) \in (-l,l) \times (0,T) \\ &u(\pm l, t) = 0, \quad t > 0 \\ &u(x,0) = u_0(x), \quad x \in (-l,l) \end{aligned}$$

near blow-up point using self-similar solution technique and observed the boundary-layer phenomena.

This fact suggest a similar result for the solution $u(x,t)$ of (P). In this paper, we prove that the solution $u(x,t)$ of (P) satisfies

$$(3) \qquad \begin{aligned} &\lim_{t \to T^-} (T-t)e^{u(0,t)} = 1, \\ &\lim_{t \to T^-} (T-t)e^{u(\alpha,t)} = 1 \end{aligned}$$

for any $\alpha \in (-a,a)$ and observe the asymptotic solution of (P).

Our method is based on their general ideas i.e., maximum principle, comparison method and self-similar solution technique.

2. Preliminaries

In this section, we set up some definitions and the auxiliary estimate of the solution of (P).

DEFINITION 2.1. A point $x \in (-a,a)$ is called a *blow-up point* if there exists a sequence $\{(x_n, t_n)\}$ such that $x_n \to x$, $t_n \to T$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$.

LEMMA 2.2 (THE MAXIMUM PRINCIPLE). *Let Ω be a bounded domain in $\mathbb{R}^n$ and let*

$$Lu \equiv \Delta u + b(x,t)\cdot\nabla u + c(x,t)u - u_t$$

in an $(n+1)$-dimensional domain $\Omega\times(0,T)$, where $b=(b_1,b_2,\cdots,b_n)$. Assume that coefficients of L are continuous functions in $\Omega\times(0,T)$ and $c(x,t)\le 0$. Suppose that either $Lu>0$ in $\Omega\times(0,T)$ or that $Lu\ge 0$ and $c(x,t)<0$. Then u cannot have a positive maximum in $\Omega\times(0,T)$.

Proof. See Friedman [2] and Protter and Weinberger [6].

THEOREM 2.3. *Suppose that $u(x,t)$ is a solution of (P) satisfying (Assumption B). For any $\eta\in(0,min\{a,T\})$ there exist a $\xi>0$ such that*

$$u_t \ge \xi e^{u(x,t)} \tag{4}$$

for $(x,t)\in(-a+\eta,a-\eta)\times(\eta,T)$.

Proof. We know that $u_t>0$ on $(-a+\eta,a-\eta)\times(\eta,T)$ by the maximum principle. Define $J(x,t)=u_t(x,t)-\xi e^{u(x,t)}$ where $\xi>0$ is to be determined. Then we observe that J satisfies

$$\begin{aligned} J_t - J_{xx} &= e^{u(0,t)}u_t(0,t) - \xi e^{\{u(0,t)+u(x,t)\}} + \xi e^{u(x,t)}u_x^2 \\ &\ge e^{u(0,t)}u_t(0,t) - \xi e^{\{u(0,t)+u(x,t)\}} \\ &\ge e^{u(x,t)}\{u_t(0,t) - \xi e^{u(0,t)}\} = e^{u(x,t)}J(0,t). \end{aligned} \tag{5}$$

Since $x=0$ is the blow-up point, if η is sufficiently small, $e^{u(x,t)}<C_1<\infty$ in $\partial(-a+\eta,a-\eta)\times(\eta,T)$. Also, $u_t\ge C_2>0$ on the parabolic boundary of $(-a+\eta,a-\eta)\times(\eta,T)$. For $\xi>0$ sufficiently small, we have

$$J = u_t - \xi e^{u(x,t)} \ge C_2 - \xi C_1 > 0$$

on the parabolic boundary $(-a+\eta,a-\eta)\times(\eta,T)$. Hence, $u_t\ge\xi e^{u(x,t)}$ for $(x,t)\in(-a+\eta,a-\eta)\times(\eta,T)$.

COROLLARY 2.4. *Suppose that $u(x,t)$ is a solution of (P) satisfying (Assumption B). For any $\eta \in (0, min\{a,T\})$ there exists M such that*

$$u(x,t) \leq M - \log(T-t), \tag{6}$$

for $(x,t) \in (-a+\eta, a-\eta) \times (\eta, T)$.

Proof. By Theorem 2.3, integrating (4) from t to T, we obtain that $u(x,t) \leq M - \log(T-t)$ in $(-a+\eta, a-\eta) \times (\eta, T)$.

THEOREM 2.5. *Let $u(x,t)$ be a solution of (P) and let for $t_0 \in (0,T)$, $u_* = max_{x\in[-a,a]} u(x,t_0)$. Define $\tilde{f}(u) = \int_0^u f(u)du$ for $f(u) = e^u$. Then if $\tilde{f}(u_*) \geq \tilde{f}(u_0(x)) + \frac{1}{2}u_0{}'(x)^2$, then we have*

$$\frac{1}{2}u_x^2 \leq e^{u(0,t_0)} \tag{7}$$

in $(-a,a) \times (0,T)$, if t_0 is near T.

Proof. Since our proof is similar to the proof of Theorem 3.1 in [3], we omit it.

REMARK. Since our concern is the asymptotic behavior near blow-up point, we have no interest in the behavior of the solution in the neighborhood of the parabolic boundary of $(-a,a)$.

3. Main Results

To analyze the blow-up behavior of $u(x,t)$ near $x = 0$ and $t = T$, we introduce the rescaled solution

$$w(y,s) = (T-t)e^{u(x,t)} \tag{8}$$

with

$$x = (T-t)^{\frac{1}{2}}y, \ T - t = e^{-s}. \tag{9}$$

If $x = 0$ is not a blow-up point, then $w(y,s)$ converges rapidly to zero as $s \to \infty$.

If u solves (P), then w solves

$$(10) \qquad w_s - w_{yy} + \frac{1}{2} y w_y + \frac{w_y^2}{w} - (w(0,s)-1)w(y,s) = 0.$$

in the domain $W_0 \equiv \{(y,s) \in \mathbb{R}^2 \mid s > s_0 = -\log T, y \in (-a e^{\frac{s}{2}}, a e^{\frac{s}{2}})\}$.

Notice that the solution $w(y,s)$ clearly exists for all $s > s_0$ and $w(y,s)$ is always strictly positive in W_0.

THEOREM 3.1. *If w is defined by (8) and (9), then*

$$0 < w(y,s) \le e^M,$$
$$\frac{w_y(y,s)}{w(y,s)} \le M'(M).$$

Proof. Immediately follows by Corollary 2.4 and Theorem 2.5.

To derive energy identities for w, multiplying (10) with ρw or ρw_s and then integrating by parts, they involve the "energy"

$$E[w](s) = \frac{1}{2} \int \rho \frac{w_y^2}{w^2} dy - \int \rho w dy + \int \rho \log w dy$$

where $\rho(y) = e^{-\frac{y^2}{4}}$ and the domain of integration being $(-a e^{\frac{s}{2}}, a e^{\frac{s}{2}})$.

Also, we consider the stationary equation of (10)

$$(11) \qquad w_{yy} - \frac{1}{2} y w_y - \frac{w_y^2}{w} + (w(0)-1)w(y) = 0, \quad y \in \mathbb{R}^1.$$

Using the same technique in [4], we can observe the following lemma.

LEMMA 3.2. *If $w(y,s)$ is a solution of (10), then $w(y,s)$ has a limit, independent of any choise of subsequence, as $s \to \infty$ and the limit is the solution $w(y)$ of (11).*

Let $v(y) = \log w(y)$. Then

$$(12) \qquad \begin{aligned} v &\le M, \\ v_y &\le M'(M) \end{aligned}$$

by Theorem 3.1.

Also, we observe that if w solves the equation (11) then v solves

$$v_{yy} - \frac{1}{2}yv_y + e^{v(0)} - 1 = 0. \tag{13}$$

Let $y = \sqrt{2}z$. Then $v_{zz} - zv_z + 2(e^{v(0)} - 1) = 0$. Since $z = 0$ is a ordinary point in above equation, we obtain the series solution of the equation (13):

$$\begin{aligned} v(y) = c_0 + c_1\{\frac{y}{\sqrt{2}} + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n+1)!}(\frac{y}{\sqrt{2}})^{2n+1}\} \\ + c_2\{(\frac{y}{\sqrt{2}})^2 + 2\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{(2n+2)!}(\frac{y}{\sqrt{2}})^{2n+2}\} \end{aligned} \tag{14}$$

where c_0 and c_1 are any constants, and $c_2 = 1 - e^{v(0)}$. By (12), since $v(y)$ is bounded, c_1 and c_2 must be zero.

We summarize:

THEOREM 3.3. *Suppose that $w(y)$ is the only bounded positive solution with $\frac{w_y}{w} \le M'$ of (11). Then $w(y) \equiv 1$.*

THEOREM 3.4. *If $u(x,t)$ is a solution of (P), then*

$$\lim_{t\to T^-} e^{u(0,t)}(T-t) = 1.$$

Proof. The proof is the direct result of Lemma 3.2 and Theorem 3.3.

Now, we investigate that if t is sufficiently near T, then $e^{u(x,t)}(T-t)$ approximately equals 1 in $(-a+\delta, a-\delta)$, where $\delta \to 0$ as $t \to T^-$. This is so-called *boundary-layer phenomena.*

For any $\alpha \in (-a, a)$, define the function

$$w_\alpha(y,s) = e^{u(x,t)}(T-t)$$

with

$$x-\alpha=(T-t)^{\frac{1}{2}}y,\ T-t=e^{-s}.$$

Then

$$\frac{\partial w_\alpha}{\partial s}-\frac{\partial^2 w_\alpha}{\partial y^2}+\frac{1}{2}y\frac{\partial w_\alpha}{\partial y}+\frac{(\frac{\partial w_\alpha}{\partial y})^2}{w_\alpha}-(w(0,s)-1)w_\alpha=0$$

where $w(0,s)$ is defined in (8).

THEOREM 3.5. *Suppose that* $u(x,t)$ *is a solution of* (P). *Then*

$$\lim_{t\to T^-} e^{u(\alpha,t)}(T-t)=1$$

for all $\alpha\in(-a,a)$. *That is, blow-up set for* (P) *is the whole domain* $(-a,a)$.

Proof. By Theorem 2.5 and Lemma 3.2, $w_\alpha(y,s)$ has a limit $w_\alpha(y)$, independent of any choice of subsequences, as $s\to\infty$ and the limit $w_\alpha(y)$ solves

$$\frac{\partial^2 w_\alpha}{\partial y^2}-\frac{1}{2}y\frac{\partial w_\alpha}{\partial y}-\frac{(\frac{\partial w_\alpha}{\partial y})^2}{w_\alpha}+(w(0,s)-1)w_\alpha=0$$

Similarly, we set $v_\alpha(y)=\log w_\alpha(y)$. Then $v_\alpha(y)$ satisfies

$$\frac{\partial^2 v_\alpha}{\partial y^2}-\frac{1}{2}y\frac{\partial v_\alpha}{\partial y}+(e^{v(0)}-1)=0.$$

By Theorem 3.3, $w_\alpha(y)\equiv 1$. Hence, since $y=0$ if and only if $x=\alpha$,

$$\lim_{t\to T^-} e^{u(\alpha,t)}(T-t)=1.$$

REMARK. Since α is arbitrary point in $(-a, a)$, we can see that $e^{u(x,t)-u(0,t)}$ has boundary-layer phenomena as $t \to T^-$ from Theorems 3.3 and 3.5.

To investigate the growth rate of blow-up behavior of $u(x,t)$ near blow-up point, we consider the problem

$$(15)\qquad \begin{aligned} &u_t^* = u_{xx}^* - \log(T-t), \quad (x,t) \in (-a,a)\times(0,T)\\ &u^*(\pm a, t) = 0, \quad t > 0\\ &u^*(x,0) = u_0(x), \quad x \in (-a,a). \end{aligned}$$

Since for any $\epsilon > 0$ sufficiently small, there exists $\sigma_0 > 0$ such that

$$(1-\epsilon)u^*(x,t) \le u(x,t) \le (1+\epsilon)u^*(x,t), \ t \in (\sigma_0, T),$$

we observe that if $u^*(x,t)$ is a solution of (15), then $u^*(x,t)$ approximately shows the behaviors of $u(x,t)$. Hence, we have the following asymptotic solution of (P).

THEOREM 3.6. *Suppose that $u^*(x,t)$ is a solution of (15). Then we have*

$$\begin{aligned} u^*(x,t) = &\sum_{n=1}^{\infty}(A_n cos(\frac{n\pi}{2a}x) + B_n sin(\frac{n\pi}{2a}x))e^{-(\frac{n\pi}{2a})^2 t}\\ &+ \int_0^t \{\sum_{n=1}^{\infty} C_n(\tau)cos(\frac{n\pi}{2a}x)e^{(\frac{n\pi}{2a})^2(\tau-t)}\}d\tau \end{aligned}$$

where $A_n = \frac{1}{a}\int_{-a}^{a} u_0(\zeta)cos(\frac{n\pi}{2a}\zeta)d\zeta$, $B_n = \frac{1}{a}\int_{-a}^{a} u_0(\zeta)sin(\frac{n\pi}{2a}\zeta)d\zeta$ and

$$C_n(\tau) = \begin{cases} 0, & n = 4m\\ -\dfrac{8a}{n\pi}\log(T-\tau), & n = 4m+1\\ 0, & n = 4m+2\\ \dfrac{8a}{n\pi}\log(T-\tau), & n = 4m+3 \end{cases}$$

where $m \in \mathbf{N}$.

Proof. The proof of the above theorem follows from the separation of variables and manipulative calculation.

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