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ON SUBREGULAR POINTS FOR SOME CASES OF LIE ALGEBRA

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Abstract We shall define three kinds of points for algebraic varieties associated to the center $\mathfrak{Z}$ of $\mathcal{U}(L)$ which is the universal enveloping algebra of a finite-dimensional modular Lie algebra over an algebraically closed field F of prime characteristic p. We announce here that $sp_4(F)$ with $p = 2$ has a subregular point.

1. Introduction

It goes without saying that classification of simple Lie algebras and their representations is very important, not that it is simply a big problem but that it is closely related to other branches of mathematics, applied mathematics and theoretical physics in particular.

Representation theory of the finite-dimensional Lie algebra L is determined on the whole by the maximal spectrum of the center $\mathfrak{Z} = \mathfrak{Z}(\mathcal{U}(L))$ of its universal enveloping algebra $\mathcal{U}(L)$; here we are mainly dealing with an algebraically closed field F of prime characteristic p as far as we are concerned with the ground field of L.

In 1954, Zassenhaus announced that any specialization of $\mathfrak{Z}$ onto an F-algebra A decides a specialization of $\mathcal{U}(L)$ onto a finitely

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generated A-ring B, which is uniquely determined up to isomorphisms over A, showing that the classes of equivalent absolutely irreducible representations correspond in 1-1 fashion to the specializations of $\mathfrak{Z}$ into F except for a subvariety of $\mathfrak{Z}$ characterized by the vanishing of the specialized discriminant ideal of $\mathcal{U}(L)$ over $\mathfrak{Z}$ and the degree of those representations equals p^m with $[Q(\mathcal{U}(L)) : Q(\mathfrak{Z})] = p^{2m}$.

In addition, he asserted that $\mathfrak{Z}$ is just a normal algebraic variety of the same dimension as $dim_F L$ and $\mathcal{U}(L)$ becomes a maximal order of the division algebra $Q(\mathcal{U}(L)) := (\mathfrak{Z} \setminus \{0\})^{-1}\mathcal{U}(L)$ of dimension p^{2m} over the quotient field $Q(\mathfrak{Z})$ of $\mathfrak{Z}$ [44]. I would like to conjecture here that it becomes smooth for classical Lie algebras with $p > 7$. Some heuristic information appears in my recent paper [19].

In 1967, Rudakov and Shafarevich showed that there exists a 1-1 correspondence between maximal points and irreducible p-dimensional S-representations (cf. [39]) of $sl_2(F)$ provided that the point P of the manifold $Spec_m(\mathfrak{Z})$ does not equal $(0,0,0,k^2)$ with $k(\neq 0) \in F$; points $P = (0,0,0,k^2)$ with $k \neq 0$ correspond to two kinds of irreducible $P-$representa- tions of degree k and $p-k$; $(0,0,0,0)$ is none other than the irreducible $P-$representation $V(p-1)$ [26]. Of course, $p \neq 2$ in this situation. The standard basis of $sl_2(F)$ is $\{e,f,h\}$ as usual with $[f,e] = -h$, $[f,h] = 2f$, $[e,h] = -2e,$; the elements $x := f^p$, $y := e^p$, $z := h^p - h$, $t := (h+1)^2 + 4fe$ generate $\mathfrak{Z}(\mathcal{U}(sl_2(F)))$ in $\mathcal{U}(sl_2(F))$; then $Spec_m(\mathfrak{Z}(\mathcal{U}(sl_2(F))))$ is defined in $F[x,y,z,t]$ by the algebraic equation $z^2 - \prod_{i=0}^{p-1}(t-i^2) + 4xy = 0$.

Curtis and Steinberg classified $P-$representations earlier for modular simple Lie algebras leaving their dimension formula problem open. By the way, the algebraic variety $Spec_m(\mathfrak{Z})$ has three kinds of points corresponding to p^m-dimensional S- representations with $S \neq 0$ and lesser dimensional S-representations with $S \neq 0$ and P- representations respectively [39], which is also attributed to Zassenhaus [44]. In the Lie algebra literature, we could not find names given to these points ; so we called them regular points, subregular points and p-points respectively. We hope, however, to have better names than these.

In 1988, Helmut Strade and R.Farnsteiner investigated spectra for $\mathcal{U}(L)$ very well in their recent book [39], but they did not mention such ingredients as are necessary for modular representation theory.

In this paper, we exhibit some examples showing that there may be subregular points for $L = sp_4(F)$ with $p = 2$ even though there isn't any such point for $L = sp_4(F)$ with $p > 2$.

2. Exact definition of 3 kinds of points

Let F be an algebraically closed field of prime characteristic and L a finite dimensional restricted Lie algebra with basis $\{x_i | 1 \leq i \leq n\}$. Further let $\mathcal{O}(L)$ be the $alg_F \langle \{x_i^p - x_i^{[p]}\} \cup \mathfrak{Z}(L) \rangle$ in $\mathcal{U}(L)$ with $\mathfrak{Z}(L)$ center of L ; then $\mathcal{O}(L)$ becomes the Noether normalization of $\mathfrak{Z}$, so that $\exists s_i \in \mathfrak{Z}, 1 \leq i \leq n'$ such that they are integral over $\mathcal{O}(L)$ and $\mathfrak{Z} = \mathcal{O}(L)[s_1, \cdots, s_{n'}]$. Let $h : \mathcal{O}(L)[X_1, \cdots, X_{n'}] \to \mathfrak{Z}$ be the evaluation (algebra) homomorphism sending $X_i \mapsto s_i$ for $1 \leq i \leq n'$; then we have $\mathfrak{Z}(\mathcal{U}(L)) = \mathcal{O}(L)[s_1, \cdots, s_{n'}] \cong \mathcal{O}(L)[X_1, \cdots, X_{n'}]/Ker\ h$ which becomes a coordinate ring on a normal algebraic variety $V(Ker\ h)$ of degree n [44]. Hence any maximal ideal of $\mathfrak{Z}(\mathcal{U}(L)) =: \mathfrak{Z}$ may be represented by a coordinate $(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{n'})$, where η_i's are roots of $Ker\ h$ for independent variables ξ_j's $(1 \leq j \leq n)$ corresponding to variables $x_j^p - x_j^{[p]}$.

Now following Zassenhaus, we have a mapping φ which goes from the set of all finite dimensional irreducible L-modules onto $Spec_m(\mathfrak{Z})$ which is the set of maximal ideals of $\mathfrak{Z}$. Here we may define 3 kinds of points in this spectrum as follows : we call $(0, \cdots, 0, \eta_1, \cdots, \eta_{n'})$ a *$P-$point* since it gives rise to $P-$representations ; the point $(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{n'})$ with $dim_F(\mathcal{U}(L)/m_j) = p^{2m}$ gives rise to p^m-dimensional $S-$representation $(S \neq 0)$, where m_j is a maximal 2-sided ideal containing the ideal $\sum_{j=1}^{n} \mathcal{U}(L)(x_j^p - x_j^{[p]} - \xi_j) + \sum_{i=1}^{n'} \mathcal{U}(L)(s_i - \eta_i)$ with ξ_j's and η_i's in F satisfying $Ker\ h$ if they replace $x_j^p - x_j^{[p]}$ and s_i's respectively, so that we call the point $(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{n'})$ a *regular point*

; the rest case gives rise to $S-$representation ($S \neq 0$) module of dimension $< p^m$, so that the point is called a *subregular point*.

3. $Irr(s, \mathcal{O}(L))$ for $L = sp_4(F)$

In the sequel, we shall fix $L = sp_4(F)$ over an algebraically closed field F of characteristic $p > 2$ unless otherwise specified ; we denote by E_{ij} an elementary matrix whose $(i,j)-$th entry is 1 with all others zero. A standard basis of L then consists of the followings : $h_1 := diag(1,0,-1,0)$, $h_2 := diag(0,1,0,-1)$, $x_1 := E_{13}$, $x_2 := E_{24}$, $x_3 := E_{14} + E_{23}$, $x_4 := E_{12} - E_{43}$ and their transposes.

Recently we have found that $Ker\ h$ becomes a principal ideal related to these elements, i.e., it becomes a hypersurface in the affine space $F^{n+n'+1}$; we now state some important facts without proofs. See [25] for more detail.

PROPOSITION 3.1. *Let s be an element in $\mathcal{U}(L)$ of the form $s := (h_1+1)^2+(h_2+1)^2+2h_1+4({}^tx_1x_1+{}^tx_2x_2)+2({}^tx_3x_3+{}^tx_4x_4)$; then (i)$s \in \mathfrak{Z}$ and (ii)$\mathfrak{Z} = \mathcal{O}(L)[s]$ in $\mathcal{U}(L)$.*

PROPOSITION 3.2. *We denote the irreducible integral equation of s over $\mathcal{O}(L)$ by $Irr(s, \mathcal{O}(L))$; then*

(i) $Irr(s, \mathcal{O}(L))$ is obtained by expanding out

$$\begin{aligned} &N_{Q(3)}^{Q(3)(h_1,h_2)}\{s - (h_1+1)^2 - (h_2+1)^2 - 2h_1\} \\ &= N_{Q(3)}^{Q(3)(h_1,h_2)}\{4({}^tx_1x_1 + {}^tx_2x_2) + 2({}^tx_3x_3 + {}^tx_4x_4)\}, \end{aligned}$$

and its degree is p^2,

(ii) s becomes separable over $\mathcal{O}(L)$ and so over $Q(\mathcal{O}(L))$.

4. Examples of subregular points

The following facts have their origins in [12] and [25], expressing probably more about dimensions of irreducible $L-$modules. See [25] for further detail.

PROPOSITION 4.1. *A point $(\xi_1, \cdots, \xi_{10}, \eta)$ with ξ_i $(1 \le i \le 10)$ not all zero corresponds in one to one fashion to a p^4-dimensional irreducible $S-$representation and $(0, \cdots, 0, \eta)$ corresponds to $P-$representations. In other words, $(\xi_1, \cdots, \xi_{10}, \eta)$ with ξ_i not all zero is a regular point, and $(0, \cdots, 0, \eta)$ is a $P-$point.*

Now we are prepared to present some examples showing that $sp_4(F)$ with $p = 2$ has some pathological aspect for certain specified points in $Spec_m(3)$, i.e., it has some subregular points in terms of our definitions in §2. As is well-known, $L = sp_4(F)$ with $p = 2$ is not simple; nevertheless it also satisfies propositions (3.1) and (3.2), so that the dimension of irreducible $L-$modules must be $\le 2^4$ by virtue of the introduction of this paper. Suppose that for $\bar{L} = sp_4(F)/FI_4$ with $p = 2$, $\xi_7 \ne 0$ with other ξ_j's $(j = 1, 2, \cdots, \hat{7}, \cdots, 10)$ zero, where ˆ denotes caret; then any point of the form $(0, 0, \cdots, 0, \xi_7, 0, \cdots, 0, \eta)$ which satisfies $Irr(s, \mathcal{O}(L))$ becomes a subregular point. We explain why this is so. We first put $\bar{\mathfrak{m}} :=$ the left ideal of $\mathcal{U}(L)$ generated by $\{x_1^p, {}^t x_1^p, h_1^p - h_1, x_2^p, {}^t x_2^p, h_2^p - h_2, x_3^p - \xi_7, {}^t x_3^p, x_4^p, {}^t x_4^p, s - \eta\}$; we next put $\rho :=$ the left ideal of $\mathcal{U}(L)$ generated by $\{\bar{\mathfrak{m}}, h_1, h_2, x_1, {}^t x_1, x_2, {}^t x_2\}$. We then insist that $\mathcal{U}(L)/\rho$ becomes an $L-$module with $1 < dim_F \mathcal{U}(L)/\rho < p^4$ induced from an $S-$representation, i.e., the point $(0, \cdots, 0, \xi_7, 0, \cdots, 0, \eta)$ becomes a subregular point by virtue of Poincare-Birkhoff-Witt theorem and the fact that $s \equiv 0$ modulo ρ. Of course, there may be similar cases which the above remarks about subregular points apply to.

All in all, we round up the above remarks in the following

PROPOSITION 4.2. *Suppose that $\bar{L} = sp_4(F)/FI_4$ over an algebraically closed field F of characteristic $p = 2$ and that $(0, \cdots, 0, \xi_7, 0, \cdots, 0, \eta)$ with $\xi_7 \ne 0$ satisfies $Ker\ h$; then it yields an irreducible $\bar{L}-$module with its dimension > 1 and $< p^4$, i.e., the point becomes a subregular point in terms of our definition.*

REMARK. In case of $L = sp_4(F)$ with its center FI_4, we obtain a similar result as above if we put $\bar{\mathfrak{m}} :=$ the left ideal of $\mathcal{U}(L)$ generated by $\{I_4, x_1^p, {}^t x_1^p, h_1^p - h_1, x_2^p, {}^t x_2^p, h_2^p - h_2, x_3^p - \xi_7, {}^t x_3^p, x_4^p, {}^t x_4^p, s - \eta\}$.

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