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BLOW–UP ESTIMATES FOR POSITIVE SOLUTIONS OF A SEMILINEAR PARABOLIC SYSTEM

ILL-HYEON NAM

Dept. of Mathematics,
Chonnam National University, Kwangju 500-757, Korea.

1. Introduction

It is well-known that solutions of the semilinear parabolic system

$$(1) \qquad \begin{aligned} &u_t = \Delta u + v^p, \quad (x,t) \in B(0,R) \times (0,T) \\ &v_t = \Delta v + u^q, \quad (x,t) \in B(0,R) \times (0,T) \\ &u(x,t) = 0 = v(x,t), \quad (x,t) \in \partial B(0,R) \times (0,T) \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in B(0,R) \end{aligned}$$

may blow up in finite time T for sufficiently large initial data u_0 and v_0, where $B(0,R)$ is the open ball of $\mathbb{R}^n$ centered at the origin of radius R and $p > 1$, $q > 1$, $u_0, v_0 : \bar{B}(0,R) \to \mathbb{R}$ are C^2, nonnegative, bounded and vanish on $\partial B(0,R)$.

In Caristi and Mitidieri [1] and Escobedo and Herrero [2], they discussed the blow-up estimates of positive solutions of the problem (1). More recently, Huang, Mochizuki and Mukai [4] studied the global existence and nonexistence, large time behavior or life span of the solutions of the problem (1).

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In this paper, we consider the initial-boundary value problem

$$(2)\qquad \begin{aligned} &u_t = \Delta u + exp(pv), \quad (x,t) \in B(\rho) \times (0,T) \\ &v_t = \Delta v + exp(qu), \quad (x,t) \in B(\rho) \times (0,T) \\ &u(x,t) = 0 = v(x,t), \quad (x,t) \in \partial B(\rho) \times (0,T) \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in B(\rho) \end{aligned}$$

where $B(\rho)$ is the open ball of $\mathbb{R}^N$ centered at the origin of radius ρ, $T < \infty$ and $p \geq q > 1$.

Assume that $u_0(x)$ and $v_0(x)$ are nonnegative, bounded, symmetric, continuous functions and vanish on $\partial B(\rho)$. Then the followings are well-known facts for the problem(2)(See Friedman and Giga [3]).

THEOREM 1. *There exists a unique nonnegative radially symmetric solutions $u(x,t)$ and $v(x,t)$ and there is a single point blow-up for the initial-boundary value problem (2).*

REMARK. In [3], they do not described the asymptotic behavior of solutions of the initial-boundary value problem (2).

From now on we shall assume that $(0,T)$ is the maximal time interval of existence and $T < \infty$. Then

$$(3)\qquad \lim_{t \to T^-} \sup_{x \in B(\rho)} (|u(x,t)| + |v(x,t)|) = \infty.$$

When (3) holds we say that the solutions blow up in finite time. Furthermore, the blow-up is simultaneous, that is, if $\lim_{t \to T^-} u(x_*, t) = \infty$ for some $x_* \in B(\rho)$ then $\lim_{t \to T^-} v(x_*, t) = \infty$ for some $x_* \in B(\rho)$.

The purpose of this paper is to describe the asymptotic behavior of the blow-up solutions of (2) near a blow-up point as $t \to T^-$. The main result of the present paper is the extension to system (2) of the result by Friedman and Giga [3] and also we used the method of Caristi and Mitidieri [1].

LEMMA 2 (THE MAXIMUM PRINCIPLE). *Let Ω be a bounded domain in $\mathbb{R}^N$ and let*

$$Lu \equiv \Delta u + b(x,t) \cdot \nabla u + c(x,t)u - u_t$$

in an $(N+1)$-dimensional domain $\Omega\times(0,T)$, where $b = (b_1, b_2, \cdots, b_N)$. Assume that coefficients of L are continuous functions in $\Omega\times(0,T)$ and $c(x,t) \le 0$. Suppose that either $Lu > 0$ in $\Omega\times(0,T)$ or that $Lu \ge 0$ and $c(x,t) < 0$. Then u cannot have a positive maximum in $\Omega \times (0,T)$.

Proof. See Protter and Weinberger [5].

LEMMA 3. *There holds:*

$$u_r < 0, v_r < 0 \quad \text{for} \;\; (r,t) \in [0,\rho) \times (0,T), \;\; r = |x|.$$

Proof. See Lemma 1.1 of Friedman and Giga [3].

LEMMA 4. *For solutions u and v of the problem (2), the following inequalities hold:*

$$u \le \frac{p}{q}v + C_1, \;\; v \le \frac{q}{p}u + C_2$$

for some constants C_1 and C_2.

Proof. Let $J = M\frac{exp(qu)}{q} - \frac{exp(pv)}{p}$ for some $M > 1$. Then

$$J_t - \Delta J + \tilde{b} \cdot \nabla J - cJ = (M-1)exp(pv + qu) \ge 0$$

where $\tilde{b}$ and c are suitable bounded functions in $B(\rho) \times (0,T')$ for any $T' < T$.

Applying the maximum priciple, we see that $J \ge 0$ if M is sufficiently large so that $J(x,0) > 0$. Consequently, $M\frac{exp(qu)}{q} \ge \frac{exp(pv)}{p}$ implies $u \le \frac{p}{q}v + C_1$. Similarly, $v \le \frac{q}{p}u + C_2$.

THEOREM 5(THE MAIN RESULT). *Suppose that for some $\rho >$ 0 the following hypertheses are satisfed:*

(a) $u_0, v_0 : \bar{B}(\rho) \to \mathbb{R}$ are C^2, nonnegative, bounded, radial, symmatrically decreasing and vanish on $\partial B(\rho)$,

(b) u and v are the classical solutions of the problem (2) defined on $B(\rho) \times (0, T)$ with $(0, T)$ maximal time interval of existence, and $T < \infty$,

(c) $\lim_{t \to T^-} u(0, t) = \lim_{t \to T^-} v(0, t) = \infty$,

(d) for some $(x, t) \in B(\rho) \times (0, T)$, $u_t(x, t) \geq 0$ and $v_t(x, t) \geq 0$,

(e) for each $t \in (0, T)$, $u_t(\cdot, t)$ and $v_t(\cdot, t)$ achieve the maximum at $x = 0$.

If $N \leq 2$ and $p \geq q > 1$, there exist constants M_1 and M_2 such that

$$u(x, t) \leq M_1 - \frac{p+1}{p(q+1)} \log(T - t),$$

$$v(x, t) \leq M_2 - \frac{q+1}{q(p+1)} \log(T - t).$$

2. Proof of Theorem 5

For $t \in (0, T)$, we define

$$\alpha(t) = exp(u(0, t))^{\frac{1}{\eta_1}}, \quad \beta(t) = exp(v(0, t))^{\frac{1}{\eta_2}}$$

where $\eta_1 = \frac{2(p+1)}{pq-1}$ and $\eta_2 = \frac{2(q+1)}{pq-1}$. Then $\alpha(t) \to \infty$ and $\beta(t) \to \infty$ as $t \to T$.

Let

$$z(r, t) = \frac{exp(u(\frac{r}{\gamma(t)}, t))}{\gamma(t)^{\eta_1}}, \quad w(r, t) = \frac{exp(v(\frac{r}{\gamma(t)}, t))}{\gamma(t)^{\eta_2}}$$

where $\gamma(t) = \alpha(t) + \beta(t)$ and $|x| = r$ for $t \in (0, T)$ and $r \in [0, \rho\gamma(t))$.

Since $0 \leq u(x, t) \leq u(0, t)$ and $0 \leq v(x, t) \leq v(0, t)$ by assumption (c), it follows that

$$(4) \qquad \begin{aligned} 0 < z(r, t) &\leq \frac{exp(u(0, t))}{\gamma(t)^{\eta_1}} \leq 1 \\ 0 < w(r, t) &\leq \frac{exp(v(0, t))}{\gamma(t)^{\eta_2}} \leq 1. \end{aligned}$$

By assumptions (b) and (e), we have obtain that

$$
\begin{aligned}
&0 \le \Delta z(r,t) + w(r,t)^p \\
(5) \qquad &= \frac{1}{\gamma(t)^{\eta_1+2}} exp(u(\frac{r}{\gamma(t)},t))\Delta u(\frac{r}{\gamma(t)},t) + \frac{exp(pv(\frac{r}{\gamma(t)},t))}{\gamma(t)^{p\eta_2}} \\
&\le \frac{1}{\gamma(t)^{\eta_1+2}}\{\Delta u(\frac{r}{\gamma(t)},t) + exp(pv(\frac{r}{\gamma(t)},t))\} \le \frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}}.
\end{aligned}
$$

Similary,

$$
\begin{aligned}
&0 \le \Delta w(r,t) + z(r,t)^q \\
(6) \qquad &= \frac{1}{\gamma(t)^{\eta_2+2}} exp(v(\frac{r}{\gamma(t)},t))\Delta v(\frac{r}{\gamma(t)},t) + \frac{exp(qu(\frac{r}{\gamma(t)},t))}{\gamma(t)^{q\eta_1}} \\
&\le \frac{1}{\gamma(t)^{\eta_2+2}}\{\Delta v(\frac{r}{\gamma(t)},t) + exp(qu(\frac{r}{\gamma(t)},t))\} \le \frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}}.
\end{aligned}
$$

for any $t \in (0,T)$ and $r \in [0, \rho\gamma(t))$.

Using the hypotheses (c), we have that

$$
\begin{aligned}
\Delta z(r,t) + w^p(r,t) &= z_{rr} + \frac{N-1}{r} z_r + w^p(r,t), \\
\Delta w(r,t) + z^q(r,t) &= w_{rr} + \frac{N-1}{r} w_r + z^q(r,t).
\end{aligned}
$$

Hence we can rewrite (5) and (6) in the following forms:

$$
(7) \qquad
\begin{aligned}
0 \le z_{rr} + \frac{N-1}{r} z_r + w^p(r,t) \le \frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}} \\
0 \le w_{rr} + \frac{N-1}{r} z_r + z^q(r,t) \le \frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}}.
\end{aligned}
$$

Since u_t and v_t are nonnegative, we obtain that

$$
(8) \qquad
\begin{aligned}
0 \le z_{rr} + \frac{N-1}{r} z_r + w^p(r,t) \le \frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}} + \frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}} \\
0 \le w_{rr} + \frac{N-1}{r} z_r + z^q(r,t) \le \frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}} + \frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}}.
\end{aligned}
$$

for $(r,t) \in [0, \rho\gamma(t)) \times (0,T)$.

Therefore from (8) we have that

(9)

$$0 \leq (z+w)_{rr} + \frac{N-1}{r}(z+w)_r + z^p + w^p \leq 2\{\frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}} + \frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}}\}.$$

Multiplying (9) by $(z+w)_r$, we have

$$\frac{1}{2}\frac{d}{dr}\{(z+w)_r\}^2 + \frac{N-1}{r}\{(z+w)_r\}^2 + (z^q + w^p)(z+w)_r \leq 0$$

where we apply Lemma 3 in the left-side inequality of (9).

Since $\frac{N-1}{r}[(z+w)_r]^2$ is positive,

$$\frac{1}{2}\frac{d}{dr}[(z+w)_r]^2 + z^q z_r + w^p z_r + z^q w_r + w^p w_r \leq 0. \tag{10}$$

Integrating (10) from 0 to r, we get

$$\begin{aligned} &\frac{1}{2}[(z+w)_r]^2 + \frac{1}{q+1}z^{q+1} - \frac{1}{q+1}z(0,t)^{q+1} + \frac{1}{p+1}w^{p+1} \\ &- \frac{1}{p+1}w(0,t)^{p+1} + [zw^p]_0^r - p\int_0^r w^{p-1}w_r(s,t)z(s,t)ds \\ &+ [z^q w]_0^r - q\int_0^r z^{q-1}z_r(s,t)w(s,t)ds \\ &\leq 0. \end{aligned} \tag{11}$$

Since $z(r,t) \in [0,1]$, $w(r,t) \in [0,1]$, and also z_r and w_r are non-positive by Lemma 3, (11) can be reduced the following form

$$\begin{aligned} &\frac{1}{2}[(z+w)_r]^2 - \frac{1}{p+1} - \frac{1}{q+1} - 2 \\ &\qquad \leq \text{ left-side of the inequality (11) } \leq 0. \end{aligned}$$

That is,

$$\frac{1}{2}[(z+w)_r]^2 \leq \frac{1}{p+1} + \frac{1}{q+1} + 2.$$

Hence

$$|z_r + w_r| \leq \left(\frac{2}{p+1} + \frac{2}{q+1} + 4\right)^{\frac{1}{2}} \tag{12}$$

for any $(r,t) \in [0, \rho\gamma(t)) \times (0,T)$.

Now, we claim that

$$\lim_{t\to T^-} \inf\left(\frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}} + \frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}}\right) \quad \text{is positive}\,. \tag{13}$$

We prove (13) by a contradiction as in Weissler [6]. Indeed, if $\lim_{t\to T^-} \inf(\frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}} + \frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}}) = 0$, then there exists a sequence t_n in $(0,T)$ with $t_n \to T$ as $n \to \infty$ such that

$$\lim_{t\to T^-} \inf\left(\frac{u_t(0,t_n)}{\gamma(t_n)^{\eta_1+2}} + \frac{v_t(0,t_n)}{\gamma(t_n)^{\eta_2+2}}\right) = 0.$$

By (4) and (12), we know that $z(\cdot,t_n)$ and $w(\cdot,t_n)$ are equibounded and Lipschitz continuous with the Lipschitz constant less than or equal to $(\frac{2}{p+1} + \frac{2}{q+1} + 4)^{\frac{1}{2}}$.

It follows from the Arzela-Ascoli Theorem that there exists a convergent subsequence t_n(still denoted by t_n) such that

$$\begin{aligned} &z(\cdot,t_n) \to \bar{z}(\cdot) \\ &w(\cdot,t_n) \to \bar{w}(\cdot). \end{aligned} \tag{14}$$

uniformly on compact subsets of $[0,\infty)$ as $n \to \infty$.

In particular, $\bar{z}, \bar{w} \in C([0,\infty), \mathbb{R}_+)$ and because of the properties of each $z(\cdot,t_n)$ and $w(\cdot,t_n)$, we know that $\bar{z}(0) = w(0) = 1$, and $\bar{z}$ and $\bar{w}$ are nonincreasing on $[0,\infty)$. Moreover, since $\bar{z}$ and $\bar{w}$ are Lipschitz, we have that $\bar{z}$ and $\bar{w}$ are absolutely continuous on $[0,\infty)$. Considering $\bar{z}$ and $\bar{w}$ as distributions on $(0,\infty)$, clearly $z(\cdot,t_n) \to \bar{z}(\cdot)$, $w(\cdot,t_n) \to \bar{w}(\cdot)$ in the sence of distributions and hence

$$\begin{aligned} &z_r(\cdot,t_n) \to \bar{z}_r(\cdot), \quad z_{rr}(\cdot,t_n) \to \bar{z}_{rr}(\cdot) \\ &w_r(\cdot,t_n) \to \bar{w}_r(\cdot), \quad w_{rr}(\cdot,t_n) \to \bar{w}_{rr}(\cdot) \end{aligned} \tag{15}$$

as $n \to \infty$ in the sence of distributions.

From (14) and (15) it follows that

$$\begin{aligned} \bar{z}_{rr} + \frac{N-1}{r} z_r + \bar{w}^p = 0 \\ \bar{w}_{rr} + \frac{N-1}{r} w_r + \bar{z}^q = 0 \end{aligned} \tag{16}$$

as distributions on $(0, \infty)$. This can be rewritten as

$$\begin{aligned} (r^{N-1}\bar{z}_r)_r + r^{N-1}\bar{w}^p = 0 \\ (r^{N-1}\bar{w}_r)_r + r^{N-1}\bar{z}^q = 0. \end{aligned} \tag{17}$$

Since $\bar{z}$ and $\bar{w}$ are absolutely continuous, immediately using (17), we know that $\bar{z}$ and $\bar{w}$ are C^2 on $(0, \infty)$.

Further, since $\bar{z} \geq 0$ and $\bar{w} \geq 0$, the local existence and uniqueness of C^2 solutions of (16) on $(0, \infty)$ guarantees that $\bar{z} > 0$ and $\bar{w} > 0$ on $(0, \infty)$ with $\bar{z}_r(0) = \bar{w}_r(0) = 0$.

In the case $N = 1$ we easily see that a contradiction aries because $\bar{z}$ and $\bar{w}$ are strictly concave nonincreasing positive functions on $(0, \infty)$. Such a functions do not exist.

In the case $N \geq 2$ we also derive a contradiction as follows: From (17), we see that $r^{N-1}\bar{z}_r$ and $r^{N-1}\bar{w}_r$ are decreasing. Hence there exist $M < 0$ and $\sigma_0 > 0$ such that

$$\begin{aligned} r\bar{z}_r(r) < M \quad \text{for } r \in (\sigma_0, \infty) \\ r\bar{w}_r(r) < M \quad \text{for } r \in (\sigma_0, \infty). \end{aligned} \tag{18}$$

Integrating (18) from s to t with $\sigma_0 \leq s \leq t$,

$$\begin{aligned} -\bar{z}(s) < \bar{z}(t) - \bar{z}(s) < M(\log t - \log s) \\ -\bar{w}(s) < \bar{w}(t) - \bar{w}(s) < M(\log t - \log s). \end{aligned} \tag{19}$$

Letting $t \to \infty$ in (19),

$$\bar{z}(s) > -\infty \quad \textit{and} \quad \bar{w}(s) > -\infty$$

which is a contradiction to positive functions $\bar{z}$ and $\bar{w}$.

Therefore we conclude that

$$\liminf_{t\to T^-}\left(\frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}}+\frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}}\right)=K>0. \tag{20}$$

By (20), there exists $t_* \in (0,T)$ such that for any $t\in(t_*,T)$ we have

$$\begin{aligned} K &\le \frac{u_t(0,t)}{\gamma(t)^{\eta_1+2}}+\frac{v_t(0,t)}{\gamma(t)^{\eta_2+2}} \\ &\le \frac{u_t(0,t)}{exp(u(0,t))^{\frac{p(q+1)}{p+1}}}+\frac{v_t(0,t)}{exp(v(0,t))^{\frac{q(p+1)}{q+1}}} \end{aligned} \tag{21}$$

Integrating (21) on $(t,s)\subset(t_*,T)$, we obtain

$$\begin{aligned} K(s-t) \le\ & \frac{p+1}{p(q+1)}\{exp(u(0,t))\}^{\frac{-p(q+1)}{p+1}} \\ &+\frac{q+1}{q(p+1)}\{exp(v(0,t))\}^{\frac{-q(p+1)}{q+1}} \\ &-\frac{p+1}{p(q+1)}\{exp(u(0,s))\}^{\frac{-p(q+1)}{p+1}} \\ &-\frac{q+1}{q(p+1)}\{exp(v(0,s))\}^{\frac{-q(p+1)}{q+1}}. \end{aligned} \tag{22}$$

Letting $s\to T$, from (22), we get

$$K(T-t)\le\frac{p+1}{p(q+1)}\{exp(u(0,t))\}^{\frac{-p(q+1)}{p+1}}+\frac{q+1}{q(p+1)}\{exp(v(0,t))\}^{\frac{-q(p+1)}{q+1}}$$

for $t\in(t_*,T)$.

By assumption $p\ge q>1$, it follows that

$$\frac{p+1}{pq+p}\le\frac{1}{q}\quad\text{and}\quad\frac{q+1}{pq+q}\le\frac{1}{q}.$$

Hence we get

$$K(T-t)\le\frac{1}{q}[\{exp(u(0,t))\}^{\frac{-p(q+1)}{p+1}}+\{exp(v(0,t))\}^{\frac{-q(p+1)}{q+1}}].$$

Since $v(0,t) \leq \frac{q}{p}u(0,t) + \frac{1}{p}\log\frac{pM}{q}$ by Lemma 4,

$$K(T-t) \leq \frac{1}{q}[\{exp(u(0,t))\}^{\frac{-p(q+1)}{p+1}} + \{exp(\frac{q}{p}u(0,t) + \frac{1}{p}\log\frac{pM}{q})\}^{\frac{-q(p+1)}{q+1}}].$$

Choosing $M > 0$ sufficiently large so that $\{exp(\frac{1}{p}\log\frac{pM}{q})\}^{\frac{-q(p+1)}{q+1}}$ is sufficiently small, we get

$$u(x,t) \leq M_1 - \frac{p+1}{p(q+1)}\log(T-t).$$

Similary, since $u(0,t) \leq \frac{p}{q}v(0,t) + \frac{1}{q}\log\frac{qM}{p}$ by Lemma 4, we have

$$v(x,t) \leq M_2 - \frac{q+1}{q(p+1)}\log(T-t).$$

This completes the proof.

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