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Equivariant Line Bundle over a Real Representation Space

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실 표현공간 위에서의 Equivariant Line Bundle

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We show Equivariant Serre Problem in real algebraic category is true when dimension of the fiber is one i.e., we show an equivariant line bundle over a real representation space is isomorphic to a product bundle of a real representation space and some G-module.

Fiber의 dimension이 일일 때, Equivariant Serre problem이 실 대수 범주에서 맞다는 것을 보였다. 즉 실 표현공간 위에서의 equivariant line bundle은 실 표현공간과 어떤 G-module의 product bundle과 동치임을 보였다.

Key words : algebraic equivariant-vector bundle, algebraic transformation group, real representation space.

I. Introduction

Let G be a reductive algebraic group over the complex number and let B and F be G-module whose representation maps are algebraic.

In algebraic transformation groups, two outstanding problems are the following :

Equivariant Serre Problem : Is an algebraic G-vector bundle over a representation B of G trivial, i.e., isomorphic to B×F for some G-module F ?

Linearity Problem : Is an algebraic action of a reductive group G on an affine space C^n linearlizable, i.e., isomorphic to some G-module as a G-variety ?

Quillen and Suslin [8,10] showed that the Equivariant Serre Problem has an affirmative solution when G is the trivial group. Bass-Haboush [1] recognized and emphasized the connection between the Linearity Problem and the Equivariant Serre Problem. In particular, they showed that equivariant vector bundles over representations are stably trivial. Schwarz [9] made a break through in the Equivariant Serre Problem by constructing counterexamples for several infinite groups. Recently Masuda and Petrie [3,4] showed the existence of negative solutions for certain non-abelian finite groups such as dihedral groups. Also Masuda, Morser and Petre [5,6] showed that the Equivariant Serre Problem has a positive solution for an abelian group.

In this paper, we consider the above problems in the real algebraic category in line bundle case. Since there is a one to one correspondence between the family of compact real algebraic groups and that of reductive complex algebraic groups through the complexification (see [7], p 249), we take compact real algebraic groups as the acting group.

In general, it is not clear whether arguments in the complex algebraic category works in the real algebraic category because R is not algebraically closed. Now we consider the complexification of real algebraic vector bundle. The usual complexification of vector bundles is to complexify only fibers, but we complexify also base space.

Let VEC(B, F; S) be the subset of stably trivial vector bundles over the affine G-variety B depending on representations F and S of G. This is the set of isomorphism classes of G-vector bundles over B whose whitney sum with S is **F**$\oplus$**S**, where **F** and **S** denote the trivial G-vector bundle over B with a fiber which is isomorphic to F.

It is natural to ask

Complexification Problem : Let G be a compact real algebraic group and let c be the complexification map c: VEC(B, F; S)→ VEC $(B_C, F_C; S_C)$

Is the map c injective? Or weakly is the inverse image of the trivial element trivial?

If we obtain affirmative answer of the above problem, then we can get many results in real algebraic category from the complex algebraic category. If we obtain negative answer, then we get some counterexamples.

II. Main Result

Definition. $\tau : C^n \to C^n$ is an anti linear if

(i) $\tau(y_1)+\tau(y_2)=\tau(y_1+y_2)$,

for $y_1, y_2 \in C^n$.

(ii) $\tau(cy) = \bar{c}\,\tau(y)$, for $c \in C$.

Lemma 1. Let $T = \{\tau \mid \tau; C^n \to C^n$: anti linear and $\tau^2 = 1\}$. For any anti linear involution $\tau \in T$, there exists $A \in GL_n(C)$ such that $\tau = A\tau_0 A^{-1}$, where τ_0 is a standard complex conjugation.

Proof. For any $\tau \in T$, Let $V = (C^n)^\tau$ is a linear subspace of C^n over R and satisfies $V \oplus iV = C^n$. For any $v \in C^n$, v can be expressed as follows : $v = \frac{v+\tau(v)}{2} + \frac{v-\tau(v)}{2}$, where $\frac{v+\tau(v)}{2}$ is an eigenvector corresponding to eigenvalue 1 and $\frac{v-\tau(v)}{2}$ is an eigenvector corresponding to eigenvalue -1. Let $V(resp.\ V_-)$ be the eigenspace which is generated by eigenvectors corresponding to $1\ (resp. -1)$. Then $C^n = V \oplus V_-$. But there exists isomorphism between V and V_- through multiplication by i. So in fact, $V_- = iV$. Therefore, $C^n = V \oplus iV$. Now consider the map $R^n \to V$. Choose a standard basis $\{e_1, e_2, \ldots, e_n\}$ for R^n in $C^n = R^{2n}$ and a basis $\{v_1, v_2, \ldots, v_n\}$ for V in R^{2n}. Then, we obtain a $2n \times n$ matrix $(v_1 \mid v_2 \mid \cdots \mid v_n)$. Now choose a basis $\{ie_1, \ldots, ie_n\}$ for iR^n and $\{iv_1, \cdots, iv_n\}$ for iV. Then we get a $2n \times 2n$ matrix $A = \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}$, where $\left(\frac{P}{Q}\right) = (v_1 \mid v_2 \mid \cdots \mid v_n)$. This matrix A corresponds to a map between $R^n \oplus iR^n \to V \oplus iV$.

Then, there is a one-to-one correspondence between A in $GL_{2n}(R)$ and $P + iQ$ in $GL_n(C)$. So we can consider that A is in $GL_n(C)$ by the above correspondence.

Corollary 2. The map $\tau \in T$ *to* $\tau\tau_0 \in \{C \in GL_n(C) \mid C\overline{C} = 1\}$ is a bijection.

Proof. By Lemma 1, we have the following commutative diagram.

$$\{C \in GL_n(C) \mid C\overline{C} = 1\} \subset GL_n(C)$$
$$\updownarrow$$
$$S = \{u : invertible\ C-linear \mid u = \tau\tau_0, \tau \in T\}$$
$$\subset \{\ invertible\ C-linear\ \ maps\ \}$$
$$\updownarrow$$

Lemma 3. For any algebraic map $c: C^m \to \{C \in GL_n(C) \mid C\overline{C} = 1\}$ by $x \to c(x)$, $det c(x)$ is a constant. In particular, $c(x)$ is a constant if $n = 1$.

Proof. $det\ c(x)\ det\ \overline{c(x)} = 1$ implies $det\ c(x) \in C([x_1, \cdots, x_m])^* = C^*$.

Definition. $\tau: B\times F_C \to B\times F_C$ is anti linear if

(i) $\tau(x, y_1) + \tau(x, y_2) = \tau(x, y_1 + y_2)$

(ii) $\tau(x, cy) = \overline{c}\tau(x, y)$, for $c \in C$

(iii) τ is an algebraic map over R i.e., polynomial.

Corollary 4. If $\dim F_C = 1$, for any anti linear involution τ on $B \times F_C$ there exists a bundle automorphism $\Psi: B\times F_C \to B\times F_C$ such that $\tau = \Psi\tau_0\Psi^{-1}$.

Proof. First we check $\Psi\tau_0\Psi^{-1}$ is an anti linear involution.

$(\Psi\tau_0\Psi^{-1})(\Psi\tau_0\Psi^{-1}) = \Psi\tau_0^2\ \Psi^{-1}$=identity.

$$\text{and } (\Psi\tau_0\Psi^{-1})(cv) = \Psi\tau_0\ c\ (\Psi^{-1}(v)) = \Psi\overline{c}\ \tau_0(\Psi^{-1}(v)) = \overline{c}\Psi\ \tau_0(\Psi^{-1}(v)).$$

So $\Psi\tau_0\Psi^{-1}$ is an anti linear involution. Since dimension of $F_C = 1$, τ is an element of $GL_2(R)$ and $\tau^2 = 1$. Also τ is an anti linear if and only if $\tau\tau_0$ is C linear.

By the following diagram, we can define locally by analytic continuation and extend grobally. Here, since B is a representation it is simply connected.

$$b \in B$$
$$\downarrow$$
$$\tau(b)\begin{pmatrix}1&0\\0&1\end{pmatrix} \in SO(2) \cong S^1$$
$$\downarrow \log$$
$$\log\ \tau(b)\begin{pmatrix}1&0\\0&1\end{pmatrix} \in so(2) \cong R$$
$$\downarrow \exp$$
$$\exp(\frac{1}{2}\log\ \tau(b)\begin{pmatrix}1&0\\0&1\end{pmatrix}) \in SO(2) \cong U(1) \subset GL(F_C)$$

Lemma 5. Suppose that for any anti linear involution τ on $B\times F_C$ there exists a bundle automorphism $\Psi: B\times F_C \to B\times F_C$ such that $\tau = \Psi\tau_0\Psi^{-1}$.

Then, if E_c is trivial then E is trivial .

Proof. Suppose E_C is isomorphic to $B_C \times F_C$.

Then $B_C \times F_C \subset B\times F_C$ and τ acts on B trivially.

So $(B\times F_C)^\tau \cong E$

$\cong \downarrow \exists\Psi$ i. e., they are equivalent

$(B\times F_C)^{\tau_0} = B\times F$

Theorem 6. Any G-line bundle over a real representation space B is trivial.

Proof. Let c: $VEC(B, F; S) \to VEC(B_C, F_C; S_C)$ be a complexification map and E be an element of $VEC(B, F; S)$, then E_C is isomorphic to $B_C\times F_C$ by [2]. Let $\Phi: E_C \to B_C\times F_C$ be a given G-equivariant isomorphism. By corollary 4, there exists Ψ: $B\times F_C$

$\rightarrow B \times F_C$ such that $\tau = \Psi \tau_0 \Psi^{-1}$, where τ_0 is a standard complex involution. So by lemma 5, E is trivial.

Remark If X is not a real representation, then the above Theorem 6 is not true.

EXAMPLE Let E_1 be a trivial real line bundle over S^1 and E_2 be a real Hopf line bundle over S^1, then E_1 is not isomorphic to E_2 as a real vector bundle. But $E_1 \otimes C$ is ismorphic to $E_2 \otimes C$ as a complex vector bundle.

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