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PENALIZED NAVIER–STOKES EQUATIONS WITH INHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. This paper is concerned with the penalized stationary incompressible Navier–Stokes system with the inhomogeneous Dirichlet boundary condition on the part of the boundary. By taking a generalized velocity space on which the homogeneous essential boundary condition is imposed and corresponding trace space on the boundary, we pose the system to the weak form which the stress force is involved. We show the existence and convergence of the penalized system in the regular branch by extending the div-stability condition.

1. Introduction

The purpose of this paper is to show the existence and convergence of the solution for the penalized stationary Navier–Stokes equations

$$-\nu\Delta\mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon + \nabla p_\epsilon = \mathbf{f} \quad \text{in } \Omega\,, \tag{1.1}$$

and

$$\nabla \cdot \mathbf{u}_\epsilon = -\epsilon p_\epsilon \quad \text{in } \Omega \tag{1.2}$$

along with inhomogeneous Dirichlet boundary conditions on a portion of the boundary

$$\mathbf{u}_\epsilon = \begin{cases} \mathbf{0} \text{ on } \Gamma_0 \\ \mathbf{g} \text{ on } \Gamma_g\,, \end{cases} \tag{1.3}$$

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where Ω is a bounded open domain in $\mathbb{R}^n$. We suppose that the boundary $\Gamma = \partial\Omega$ is composed of two disjoint parts Γ_0 and Γ_g. Here, ν denotes the kinematic viscosity in the nondimensional form and $\mathbf{f}$ the given external body force.

The penalty method is often introduced to relax the incompressibility constraint with regard to Navier–Stokes system by introducing an artificial compressibility $-\epsilon p_\epsilon$ instead of the incompressibility constraint and to expect *near incompressibility.* For the formulation of penalty method and its numerical analysis, one may refer to [1], [3], [8] and [9]. Unlike the situations of [3] and [10], inhomogeneous boundary condition $\mathbf{g}$ is imposed on the part Γ_g of the boundary, which may induce some troublesome jumps around interface of adjacent boundary. This dynamical situations are often raisen in connection with fluid controls([4], [5] and [6]). In this paper, we will take two different settings of function spaces on the boundary, put the penalized system (1.1)–(1.3) into corresponding variational form and investigate nonlinear form of the system. The existence and convergence of the solutions of penalized system will be shown.

2. Preliminaries

Throughout this paper, $\mathcal{I}$ will be used to denote the identity mapping or the identity matrix, and C a generic constant whose value and meaning also vary with context. For Galerkin type variational formulations, we denote by $H^s(\Omega)$, the standard Sobolev space of order s with respect to the set Ω, which is the domain occupied by the flow, or its boundary Γ, or part of its boundary. For vector–valued functions and spaces, we use boldface notation, i.e., $\mathbf{H}^s(\Omega)$. We denote the inner product on $H^s(\Omega)$ or $\mathbf{H}^s(\Omega)$ by $(\cdot,\cdot)_s$ and its norm by $\|\cdot\|_s = \sqrt{(\cdot,\cdot)_s}$. Since $\mathbf{u} = \mathbf{0}$ along Γ_0, we take a generalized velocity space to be

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \,|\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}\,;$$

i.e., $\mathbf{V}$ is the space on which the *homogeneous* essential boundary condition is imposed. Let $\mathbf{V}^*$ be the dual space of $\mathbf{V}$. Note that $\mathbf{V}^*$ is a subspace of $\mathbf{H}^{-1}(\Omega)$, where the latter is the dual space of $\mathbf{H}_0^1(\Omega)$. We denote the duality between $\mathbf{V}^*$ and $\mathbf{V}$ by $< \cdot, \cdot >_{-1}$.

For the other face Γ_g of a Lipschitz continuous domain Ω, we take

$$\mathbf{L}_g^2(\Gamma) = \{\, \mathbf{s} \in \mathbf{L}^2(\Gamma) \mid \mathbf{s} = \mathbf{0} \quad \text{on } \Gamma_0 \,\}$$

and let $\gamma_g : \mathbf{V} \longrightarrow \mathbf{L}_g^2(\Gamma)$ be the trace mapping. Let us define

$$\mathbf{W} = \gamma_g(\mathbf{V})\,.$$

Let $\mathbf{W}^*$ denote its dual space and let $< \cdot, \cdot >_{-1/2,\Gamma_g}$ denote the duality between $\mathbf{W}^*$ and $\mathbf{W}$. On the other hand, we denote $\mathbf{H}^s(\Gamma_g)$ as the space of the restrictions to Γ_g of the functions in $\mathbf{H}^s(\Gamma)$ for each $s \geq 0$ and $\mathbf{H}^{-s}(\Gamma_g)$ as its dual space. It is clear that the restrictions to Γ_g of the functions of $\mathbf{W}$ is a closed subspace of $\mathbf{H}^{1/2}(\Gamma_g)$. For the given boundary force, we take

$$\mathbf{H}_0^s(\Gamma_g) = \Big\{ \boldsymbol{\phi} \in \mathbf{H}^s(\Gamma_g) \mid \text{support of } \boldsymbol{\phi} \subset \Gamma_g \text{ and } \int_{\Gamma_g} \boldsymbol{\phi} \cdot \mathbf{n}\, d\Gamma = 0 \Big\}\,.$$

We assume that the boundary force satisfy

$$\text{support of } \mathbf{g} \subset \Gamma_g \text{ and } \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{n}\, d\Gamma = 0\,. \tag{2.1}$$

This condition is necessary for the compatibility and regularity for the solutions. By the help of the condition (2.1), we assume $\mathbf{g} \in \mathbf{H}_0^s(\Gamma_g)$ for some $s \geq 1/2$. The setting of $\mathbf{W}$ and $\mathbf{g}$ avoids any troublesome jumps which may follow if one just uses $\mathbf{H}^{1/2}(\Gamma_g)$ to absorb the homogeneous boundary data. Now, let $\mathbf{s}$ be an element of $\mathbf{W}$. It is well-known that $\mathbf{W}$ is a Hilbert space with the norm

$$\|\mathbf{s}\|_{1/2,\Gamma_g} = \inf_{\mathbf{v} \in \mathbf{V}, \gamma_g \mathbf{v} = \mathbf{s}} \|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{s} \in \mathbf{W}\,.$$

Let $\mathbf{s}^*$ belong to $\mathbf{W}^*$. By the definition of the dual norm, we note that

$$\|\mathbf{s}^*\|_{-1/2,\Gamma_g} = \sup_{\mathbf{0} \neq \mathbf{s} \in \mathbf{W}} \frac{< \mathbf{s}^*, \mathbf{s} >_{-1/2,\Gamma_g}}{\|\mathbf{s}\|_{1/2,\Gamma_g}}\,.$$

In [4], we have derived the following alternate definition for the norm $\|\cdot\|_{-1/2,\Gamma_g}$ over $\mathbf{W}^*$. It will be useful in showing the existence and convergence of penalized solutions in §4.

Lemma 2.1. *It holds that*

$$\|\mathbf{s}^*\|_{-1/2,\Gamma_g} = \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}} \frac{< \mathbf{s}^*, \gamma_g \mathbf{v} >_{-1/2,\Gamma_g}}{\|\mathbf{v}\|_{1,\Omega}} \quad \forall \mathbf{s}^* \in \mathbf{W}\,. \tag{2.2}$$

We will use the quotient space

$$S = \{p \in L^2(\Omega) \mid \int_\Omega p d\Omega = 0\}$$

for the space of generalized pressures.

3. Weak variational formulation

Let $D(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$ be a symmetric tensor of first-order derivatives of $\mathbf{v}$, which represents the deformation following from the velocity $\mathbf{v}$ of the flow. We will denote the tensor product between deformation tensors by

$$D(\mathbf{u}) : D(\mathbf{v}) = \frac{1}{4}\sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) d\Omega\,.$$

We will use the two bilinear forms

$$a(\mathbf{u},\mathbf{v}) = 2\int_\Omega D(\mathbf{u}) : D(\mathbf{v})\, d\Omega \quad \forall \mathbf{u},\, \mathbf{v} \in \mathbf{H}^1(\Omega)$$

and

$$b(\mathbf{v}, q) = -\int_\Omega q\nabla\cdot\mathbf{v}\, d\Omega \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \text{ and } \forall q \in L^2(\Omega)\,,$$

and the trilinear form

$$c(\mathbf{w},\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, d\Omega \quad \forall \mathbf{u},\, \mathbf{v},\, \mathbf{w} \in \mathbf{H}^1(\Omega)\,.$$

These forms are continuous in the sense that there exist constant $C > 0$ such that

$$|a(\mathbf{u},\mathbf{v})| \le C\|\mathbf{u}\|_1\|\mathbf{v}\|_1 \quad \forall \mathbf{u},\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,,$$

$$|b(\mathbf{v},q)| \le C\|\mathbf{v}\|_1\|q\|_0 \quad \forall \mathbf{v}\in \mathbf{H}^1(\Omega) \text{ and } q\in L^2(\Omega)$$

and

$$|c(\mathbf{w},\mathbf{u},\mathbf{v})| \le C\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1 \quad \forall \mathbf{u},\, \mathbf{v},\, \mathbf{w}\in \mathbf{H}^1(\Omega)\,.$$

Moreover, we have the coercivity properties

$$a(\mathbf{v},\mathbf{v}) \ge C\|\mathbf{v}\|_1^2 \quad \forall \mathbf{v}\in \mathbf{V} \tag{3.1}$$

and the div-stability condition

$$\inf_{q\in S}\sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1\|q\|_0} \ge C\,. \tag{3.2}$$

For details concerning this forms one may consult [3], [6] and [10]. Employing this forms, the penalized equations are recast into the following particular weak form

$$\begin{aligned} \nu a(\mathbf{u}_\epsilon,\mathbf{v}) + b(\mathbf{v},p_\epsilon) + c(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,\mathbf{v}) \\ - < \mathbf{t}_\epsilon, \gamma_g\mathbf{v} >_{-1/2,\Gamma_g} = < \mathbf{f},\mathbf{v} >_{-1} \quad \forall \mathbf{v}\in\mathbf{V}\,, \end{aligned} \tag{3.3}$$

$$b(\mathbf{u}_\epsilon,q) = \epsilon(p_\epsilon,q)_0 \quad \forall q\in S \tag{3.4}$$

and

$$< \mathbf{s}^*,\mathbf{u}_\epsilon >_{-1/2,\Gamma_g} = < \mathbf{s}^*,\mathbf{g} >_{-1/2,\Gamma_g} \quad \forall \mathbf{s}^*\in\mathbf{W}^*\,. \tag{3.5}$$

Note that $\nabla\cdot((\nabla\mathbf{u}_\epsilon)^T\cdot\mathbf{v}) = \nabla\mathbf{u}_\epsilon : \nabla\mathbf{v} + \Delta\mathbf{u}_\epsilon\cdot\mathbf{v}$. Since

$$\begin{aligned} -\int_\Omega \Delta\mathbf{u}_\epsilon\cdot\mathbf{v}\,d\Omega &= \int_\Omega \nabla\mathbf{u}_\epsilon : \nabla\mathbf{v}\,d\Omega - \int_\Gamma ((\nabla\mathbf{u}_\epsilon)^T\cdot\mathbf{v})\cdot\mathbf{n}\,d\Gamma \\ &= \int_\Omega \nabla\mathbf{u}_\epsilon : \nabla\mathbf{v}\,d\Omega - \int_\Gamma ((\nabla\mathbf{u}_\epsilon)\cdot\mathbf{n})\cdot\mathbf{v}\,d\Gamma \end{aligned}$$

and $\mathbf{v}=\mathbf{0}$ along Γ_0, we obtain the formula for $\mathbf{t}_\epsilon$ over Γ_g

$$\mathbf{t}_\epsilon = -p_\epsilon\mathbf{n} + 2\nu D(\mathbf{u}_\epsilon)\cdot\mathbf{n} \tag{3.6}$$

in the distribution sense, where $\mathbf{n}$ denotes the outward unit normal vector. This represents the stress force along the inhomogeneous boundary Γ_g due to the penalized deformation.

Remark 3.1. As in [9], one may add an additional stabilization term $\frac{1}{2}\int_\Omega (\nabla \cdot \mathbf{u}_\epsilon)\mathbf{u}_\epsilon \cdot \mathbf{u}_\epsilon \, d\Omega$ to the left hand side of the equation (3.3). This extra term may be necessary for the analyticity of the penalized solutions in the sense of Cauchy–Kowalevskaya theorem. However, this extra term results in a minor change to the trilinear form $c(\cdot,\cdot,\cdot)$ in our case.

Remark 3.2. The penalty parameter ϵ in our case may be chosen in various ways. It should be chosen small enough so that the compressibility and pressure errors due to the penalization are negligible, but not so small to avoid ill-conditioning of the system. In the numerical process, its choice also depends on the dynamic viscosity and the machine precision (cf. [8]).

4. Existence and convergence of penalized solutions

In this section, we derive existence and convergence results of the penalized solution of $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{t}_\epsilon)$ to the solution $(\mathbf{u}, p, \mathbf{t})$ of the primal Navier–Stokes system. We first invoke the nonlinear functional formulation as in [2] and [3], and then we recast the system (3.3)–(3.5) into the corresponding functional setting to show the main convergence results. For the sake of completeness, we will first state the relevant results, specialized to our needs.

The abstract structure of the parameter–dependent nonlinear problems we are concerned with is of the form ;

$$F(\lambda,\psi) \equiv \psi + TG(\lambda,\psi) = 0\,, \tag{4.1}$$

where $T : Y \to X$ is a bounded linear mapping, $G : \Lambda \times X \to Y$ is a $\mathcal{C}^2$ nonlinear mapping, X and Y are Banach spaces and Λ is a compact interval of $\mathbb{R}$. Let the solution ψ of (4.1) depend on the parameter λ. We say that $\{(\lambda,\psi(\lambda)) \mid \lambda \in \Lambda\}$ is a branch of solutions of (4.1) if $\lambda \mapsto \psi(\lambda)$ is a continuous function from Λ into X such that $F(\lambda,\psi(\lambda)) = 0$. By $D_\psi F$, we denote the *Fréchet derivative* of $F(\cdot,\cdot)$ with respect to the second variable. If $D_\psi F(\lambda,\psi(\lambda))$ is an isomorphism from X into X for all $\lambda \in \Lambda$, then the branch $\lambda \mapsto \psi(\lambda)$ is called a *regular branch*. Note that $D_\psi F(\lambda,\psi) = \mathcal{I} + TD_\psi G(\lambda,\psi)$ from (4.1). Hence, if we consider $D_\psi G(\lambda,\cdot)$ as a bounded linear mapping from X into Z, a subspace of

Y, where the inclusion $Z \subset Y$ is a continuous embedding and $T|_Z : Z \to X$ is compact, then $D_\psi F$ appears to be a compact perturbation of the identity. Approximations are defined by introducing a subspace X^h of X and an approximating operator $T^h \in \mathcal{L}(Y, X^h)$, where $\mathcal{L}$ denotes the bounded linear operators between Banach spaces. The approximation problem corresponding to the nonlinear form (4.1) is to seek $\psi^h \in X^h$ such that

$$F^h(\lambda, \psi^h) \equiv \psi^h + T^h G(\lambda, \psi^h) = 0 \,. \tag{4.2}$$

The convergence to a regular branch of solutions of the approximation problem (4.2) is ensured under the following three assumptions;

$$D_\psi G(\lambda, \psi) \in \mathcal{L}(X, Z) \quad \forall \lambda \in \Lambda \quad \text{and} \quad \psi \in X \,, \tag{4.3}$$

$$\lim_{h \to 0} \|(T^h - T)y\|_X = 0 \quad \forall y \in Y \tag{4.4}$$

and

$$\lim_{h \to 0} \|(T^h - T)\|_{\mathcal{L}(Z,X)} = 0 \,. \tag{4.5}$$

Concerning approximations of the regular branch, we can now state the following fundamental result that will be used in the sequel. For the proof, refer to [2] and [3].

THEOREM 4.1. *Assume that $G : \Lambda \times X \to Y$ is a $\mathcal{C}^2$ nonlinear mapping and that the second Fréchet derivative $D_{\psi\psi} G$ is bounded on all bounded sets of $\Lambda \times X$. Assume that (4.3)–(4.5) hold and that $\{(\lambda, \psi(\lambda)) \mid \lambda \in \Lambda\}$ is a branch of regular solutions of (4.1). Then, there exists a neighborhood $\mathcal{O}$ of the origin in X and, for $h \le h_0$ small enough, a unique $\mathcal{C}^2$ function $\lambda \in \Lambda \mapsto \psi^h(\lambda) \in X^h$ such that $\{(\lambda, \psi^h(\lambda)) \mid \lambda \in \Lambda\}$ is a branch of regular solutions of (4.2) and $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover, there exists a positive constant C, independent of h and λ, such that*

$$\|\psi^h(\lambda) - \psi(\lambda)\|_X \le C \|(T^h - T) G(\lambda, \psi(\lambda))\|_X \quad \forall \lambda \in \Lambda \,. \tag{4.6}$$

The stationary Navier–Stokes equations can be posed in the nonlinear form via the Stokes operator and the parameter $\lambda = \dfrac{1}{\nu}$ = Reynold number Re. The fundamental idea of a regular branch in the study of solutions of the stationary Navier–Stokes equations is based on the fact that bifurcation points and turning points are quite rare (cf. [11]).

We can apply this structure to the penalized Navier–Stokes system (1.1)–(1.3) for the study of the convergence when ϵ tends to 0. We take $\mathbf{X} = \mathbf{V} \times S \times \mathbf{W}^*$, $\mathbf{Y} = \mathbf{V}^* \times \mathbf{W}$ and $\mathbf{Z} = \mathbf{L}^{3/2}(\Omega) \times \{\mathbf{0}\}$. We also take $\mathbf{X}^h = \mathbf{X}$ in the above discussion. For the parameter, we take $\lambda = \dfrac{1}{\nu} \in \Lambda \subset \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the nonnegative real numbers and Λ a compact interval in $\mathbb{R}^+ - \{0\}$. We define the solution operator $T \in \mathcal{L}(\mathbf{Y}; \mathbf{X})$ for the Stokes problem with inhomogeneous boundary conditions by $T(\widehat{\mathbf{f}}, \widehat{\mathbf{g}}) = (\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{t}})$ if and only if

$$
\begin{aligned}
a(\widehat{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \widehat{p}) - < \widehat{\mathbf{t}}, \gamma_g \mathbf{v} >_{-1/2,\Gamma_g} &= < \widehat{\mathbf{f}}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{V}, \\
b(\widehat{\mathbf{u}}, q) &= 0 \quad \forall q \in S, \\
< \mathbf{s}^*, \gamma_g \widehat{\mathbf{u}} >_{-1/2,\Gamma_g} &= < \mathbf{s}^*, \widehat{\mathbf{g}} >_{-1/2,\Gamma_g} \quad \forall \mathbf{s}^* \in \mathbf{W}^*.
\end{aligned}
\tag{4.7}
$$

The nonlinearity of the Navier–Stokes equations is taken into account by the mapping $G : \Lambda \times \mathbf{X} \to \mathbf{Y}$ ($(\lambda, (\mathbf{w}, q, \boldsymbol{\tau})) \mapsto (\boldsymbol{\eta}, \boldsymbol{\kappa})$) defined by

$$
\begin{aligned}
< \boldsymbol{\eta}, \mathbf{v} >_{-1} &= \lambda\, c(\mathbf{w}, \mathbf{w}, \mathbf{v}) - \lambda < \mathbf{f}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{V}, \\
< \mathbf{s}^*, \boldsymbol{\kappa} >_{-1/2,\Gamma_g} &= - < \mathbf{s}^*, \mathbf{g} >_{-1/2,\Gamma_g} \quad \forall \mathbf{s}^* \in \mathbf{W}^*,
\end{aligned}
\tag{4.8}
$$

where $(\mathbf{f}, \mathbf{g})$ is given in $\mathbf{V}^* \times \mathbf{W}$.

Since the weak formulation of the Navier–Stokes equations can be written by

$$
\begin{aligned}
a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \lambda p) - < \lambda \mathbf{t}, \gamma_g \mathbf{v} >_{-1/2,\Gamma_g} & \\
= - [\, \lambda c\,(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda < \mathbf{f}, \mathbf{v} >_{-1}] & \quad \forall\, \mathbf{v} \in \mathbf{V}, \\
b(\mathbf{u}, \lambda q) = 0 & \quad \forall q \in S, \\
< \mathbf{s}^*, \gamma_g \mathbf{u} >_{-1/2,\Gamma_g} = - [\, - < \mathbf{s}^*, \mathbf{g} >_{-1/2,\Gamma_g}] & \quad \forall \mathbf{s}^* \in \mathbf{W}^*,
\end{aligned}
\tag{4.9}
$$

and the mapping G corresponds to the weak formulation of

$$
\begin{cases}
\boldsymbol{\eta} = \lambda (\mathbf{w} \cdot \nabla) \mathbf{w} - \lambda\, \mathbf{f}, \\
\boldsymbol{\kappa} = - \mathbf{g}.
\end{cases}
$$

Substituting $\mathbf{w} = \mathbf{u}$, we obtain from (4.9) that $q = \lambda p$, $\boldsymbol{\tau} = \lambda \mathbf{t}$ and $(\mathbf{u}, \lambda p, \lambda \mathbf{t}) = -TG(\mathbf{u}, \lambda p, \lambda \mathbf{t})$. Hence, we have

$$(\mathbf{u}, \lambda p, \lambda \mathbf{t}) + TG(\lambda, (\mathbf{u}, \lambda p, \lambda \mathbf{t})) = 0 \tag{4.10}$$

which is equivalent to the weak variational form (4.9) of the primal stationary incompressible Navier–Stokes equations.

Next, we associate $T^\epsilon : \mathbf{Y} \to \mathbf{X}$ ($(\widehat{\mathbf{f}}, \widehat{\mathbf{g}}) \mapsto (\widehat{\mathbf{u}}_\epsilon, \widehat{p}_\epsilon, \widehat{\mathbf{t}}_\epsilon)$) with the penalized Stokes operator defined by

$$\begin{aligned} a(\widehat{\mathbf{u}}_\epsilon, \mathbf{v}) + b(\mathbf{v}, \widehat{p}_\epsilon) - < \widehat{\mathbf{t}}_\epsilon, \gamma_g \mathbf{v} >_{-1/2,\Gamma_g} = < \widehat{\mathbf{f}}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{V}\,, \\ b(\widehat{\mathbf{u}}_\epsilon, q) = \epsilon\, (\widehat{p}_\epsilon, q)_0 \quad \forall q \in S\,, \\ < \mathbf{s}^*, \gamma_g \widehat{\mathbf{u}}_\epsilon >_{-1/2,\Gamma_g} = < \mathbf{s}^*, \widehat{\mathbf{g}} >_{-1/2,\Gamma_g} \quad \forall \mathbf{s}^* \in \mathbf{W}^*\,. \end{aligned} \tag{4.11}$$

Then, the penalized Navier–Stokes equations (1.1)–(1.3) is equivalent to

$$(\mathbf{u}_\epsilon, \lambda p_\epsilon, \lambda \mathbf{t}_\epsilon) + T^\epsilon G(\lambda, (\mathbf{u}_\epsilon, \lambda p_\epsilon, \lambda \mathbf{t}_\epsilon)) = 0\,. \tag{4.12}$$

Now, let us turn to the existence of solutions for the specified systems (4.7) and (4.11). We need to extend the div-stability condition by coupling the presure and the stress force on Γ_g together.

LEMMA 4.2. *For every $(q, \mathbf{s}^*) \in S \times \mathbf{W}^*$, there exists a positive constant C such that*

$$C \| (q, \mathbf{s}^*) \|_{\mathbf{S} \times \mathbf{W}^*} \le \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q) - < \mathbf{s}^*, \mathbf{v} >_{-1/2,\Gamma_g}}{\|\mathbf{v}\|_1}\,. \tag{4.13}$$

Proof. Let $(q, \mathbf{s}^*) \in S \times \mathbf{W}^*$ be given. By applying Riesz representation theorem, given $\mathbf{s}^* \in \mathbf{W}^*$ one can choose an $\mathbf{s} \in \mathbf{W}$ such that $\|\mathbf{s}\|_{1/2,\Gamma_g} = \|\mathbf{s}^*\|_{-1/2,\Gamma_g}$ and $< \mathbf{s}^*, \boldsymbol{\eta} >_{-1/2,\Gamma_g} = \int_{\Gamma_g} \mathbf{s} \cdot \boldsymbol{\eta}\, d\Gamma \quad \forall \boldsymbol{\eta} \in \mathbf{W}$. Next, one may choose a $\mathbf{v} \in \mathbf{V}$ such that

$$\begin{cases} \nabla \cdot \mathbf{v} = q & \text{in } \Omega\,, \\ \mathbf{v} = -\mathbf{s} & \text{on } \Gamma_g\,. \end{cases} \tag{4.14}$$

Since $q \in S$, (4.14) is wellposed (cf. [8]) and it follows that

$$\|\mathbf{v}\|_1 \leq C_1(\|q\|_0 + \|\mathbf{s}\|_{1/2,\Gamma_g}) = C_1(\|q\|_0 + \|\mathbf{s}^*\|_{-1/2,\Gamma_g})$$

for some positive constant C_1. For this choice of $\mathbf{v}$ and $\mathbf{s}$, we have

$$\begin{aligned} b(\mathbf{v},q) - <\mathbf{s}^*, \gamma_g \mathbf{v}>_{-1/2,\Gamma_g} &= \int_\Omega (\nabla \cdot \mathbf{v}) q \, d\Omega - \int_{\Gamma_g} \mathbf{s} \cdot (\gamma_g \mathbf{v}) \, d\Gamma \\ &= \|q\|_0^2 + \|\mathbf{s}\|_{1/2,\Gamma_g}^2 = \|q\|_0^2 + \|\mathbf{s}^*\|_{-1/2,\Gamma_g}^2 \\ &\geq \frac{1}{2}(\|q\|_0 + \|\mathbf{s}^*\|_{-1/2,\Gamma_g})^2 \geq \frac{1}{2C_1}(\|q\|_0 + \|\mathbf{s}^*\|_{-1/2,\Gamma_g})\|\mathbf{v}\|_1 \\ &\geq C(\|q\|_0 + \|\mathbf{s}^*\|_{-1/2,\Gamma_g})\|\mathbf{v}\|_1 = C\|(q,\mathbf{s}^*)\|_{S \times \mathbf{W}^*} . \end{aligned}$$

Here, positive constant C was taken so that $0 < C < \dfrac{1}{2C_1}$. □ □

The existence of the solutions of the system (4.7) and (4.11) depends on the following.

LEMMA 4.3. ([8]) *Let $\mathfrak{X}$ and $\mathfrak{M}$ be two Hilbert spaces. Let $\mathfrak{A}(\cdot,\cdot)$ be a bounded bilinear form on $\mathfrak{X}$ and $\mathfrak{B}(\cdot,\cdot)$ a bounded bilinear form on $\mathfrak{X} \times \mathfrak{M}$. Define $\mathfrak{Z} = \{u \in \mathfrak{X} \mid \mathfrak{B}(u,q) = 0, \text{ for all } q \in \mathfrak{M}\}$. Assume that for some positive constants α and β*

$$\mathfrak{A}(z,z) \geq \alpha \|z\|_{\mathfrak{X}}^2 \quad \forall z \in \mathfrak{Z} \tag{4.15}$$

and

$$\inf_{0 \neq q \in \mathfrak{M}} \sup_{0 \neq u \in \mathfrak{X}} \frac{\mathfrak{B}(u,q)}{\|u\|_{\mathfrak{X}} \|q\|_{\mathfrak{M}}} \geq \beta . \tag{4.16}$$

Then, for any given $(f,\psi) \in (\mathfrak{X} \times \mathfrak{M})^*$, there exists a unique $(u,p) \in \mathfrak{X} \times \mathfrak{M}$ such that

$$\mathfrak{A}(u,v) + \mathfrak{B}(v,p) = < f, v > \quad \forall\, v \in \mathfrak{X} \tag{4.17}$$

and

$$\mathfrak{B}(u,q) = < \psi, q > \quad \forall\, q \in \mathfrak{M} . \tag{4.18}$$

THEOREM 4.4. *Suppose* $b(\cdot,\cdot)$ *satisfy the div-stability condition. Then, given* $(\widehat{\mathbf{f}},\widehat{\mathbf{g}}) \in \mathbf{V}^* \times \mathbf{W}$, *there exists unique solutions* $(\widehat{\mathbf{u}},\widehat{p},\widehat{\mathbf{t}})$ *and* $(\widehat{\mathbf{u}}_\epsilon,\widehat{p}_\epsilon,\widehat{\mathbf{t}}_\epsilon)$ *in* $\mathbf{V}\times S\times \mathbf{W}^*$ *of the systems (4.7) and (4.11), respectively.*

Proof. Let $\mathcal{F} : \mathbf{V} \to S\times \mathbf{W}^*$ be the bounded linear operator defined by

$$(\mathcal{F}\mathbf{v},(q,\mathbf{s}^*))_{S\times\mathbf{W}^*} = b(\mathbf{v},q) - < \mathbf{s}^*, \gamma_g\mathbf{v} >_{-1/2,\Gamma_g} \ .$$

Then, (4.11) can be rewritten by for all $\mathbf{v}\in\mathbf{V}$ and $(q,\mathbf{s}^*)\in S\times\mathbf{W}^*$,

$$\begin{aligned} a(\widehat{\mathbf{u}}_\epsilon,\mathbf{v}) + (\mathcal{F}\mathbf{v},(\widehat{p}_\epsilon,\widehat{\mathbf{t}}_\epsilon)) &= < \widehat{\mathbf{f}},\mathbf{v} >_{-1}\,, \\ (\mathcal{F}\widehat{\mathbf{u}}_\epsilon,(q,\mathbf{s}^*)) &= \epsilon(\widehat{p},q)_0 - < \mathbf{s}^*,\widehat{\mathbf{g}} >_{-1/2,\Gamma_g} \ . \end{aligned} \tag{4.19}$$

Now, putting $\mathfrak{X}=\mathbf{V}$, $\mathfrak{M}=S\times\mathbf{W}^*$, $\mathfrak{A}(\cdot,\cdot)=a(\cdot,\cdot)$ and $\mathfrak{B}(\mathbf{v},(q,\mathbf{s}^*))$ into $(\mathcal{F}\mathbf{v},(q,\mathbf{s}^*))$, then we may pose (4.19) into the same situation with that of Lemma 4.3. Since $\mathbf{V}\ni\mathbf{v}=\mathbf{0}$ on Γ_0, we can easily verify that $\ker\mathcal{F}\subset\mathbf{H}^1_0(\Omega)$. Since $\sqrt{a(\cdot,\cdot)}$ over $\mathbf{V}$ is equivalent to $\|\cdot\|_1$ by Korn's Lemma, $a(\cdot,\cdot)$ is coercive over $\ker\mathcal{F}$. So, condition (4.15) is satisfied. Condition (4.16) is also satisfied by Lemma 4.2. Hence, combined with Lemma 4.2 with Lemma 4.3, (4.19) has a unique solution $(\widehat{\mathbf{u}}_\epsilon,(\widehat{p}_\epsilon,\widehat{\mathbf{t}}_\epsilon))\in\mathbf{V}\times(S\times\mathbf{W}^*)$. Similarly, we can show that (4.7) has also a unique solution $(\widehat{\mathbf{u}},\widehat{p},\widehat{\mathbf{t}})$. □ □

The following estimates follow from the wellposedness of the systems (4.11) and (4.7):

$$\begin{aligned} \|\widehat{\mathbf{u}}_\epsilon\|_1 + \|\widehat{p}_\epsilon\|_0 + \|\widehat{\mathbf{t}}_\epsilon\|_{-1/2,\Gamma_g} &\le C(\|\widehat{\mathbf{f}}\|_{-1} + \|\widehat{\mathbf{g}}\|_{1/2,\Gamma_g})\,, \\ \|\widehat{\mathbf{u}}\|_1 + \|\widehat{p}\|_0 + \|\widehat{\mathbf{t}}\|_{-1/2,\Gamma_g} &\le C(\|\widehat{\mathbf{f}}\|_{-1} + \|\widehat{\mathbf{g}}\|_{1/2,\Gamma_g})\,. \end{aligned}$$

In the penalized system corresponding to (4.7), Lagrange multipliers may be employed to relax constraints of the incompressibility as well as the inhomogeneous boundary condition. Let us concern with the following saddle point problem:

$$\inf_{\mathbf{s}^*\in\mathbf{W}^*}\sup_{\mathbf{v}\in\mathbf{V}} \mathfrak{E}(\mathbf{v},\mathbf{s}^*)\,,$$

where $\mathfrak{E} : \mathbf{V} \times \mathbf{W}^* \to \mathbb{R}$ is a Lagrangian difined by

$$\mathfrak{E}(\mathbf{v}, \mathbf{s}^*) = 2\nu \int_\Omega D(\mathbf{v}) : D(\mathbf{v})\, d\Omega + \frac{1}{2\epsilon} \int_\Omega (\nabla \cdot \mathbf{v})^2\, d\Omega - \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega - \int_{\Gamma_g} \mathbf{s}^* \cdot (\gamma_g \mathbf{v} - \mathbf{g})\, d\Gamma\,.$$

The coercivity of $a(\cdot, \cdot)$ and the condition (4.13) may guarantee the existence and uniqueness of the saddle point of the quadratic form $\mathfrak{E}$. The saddle point $(\mathbf{u}_\epsilon, \mathbf{t}_\epsilon)$ of $\mathfrak{E}$ satisfies

$$\begin{aligned} \nu \int_\Omega D(\mathbf{u}_\epsilon) : D(\mathbf{v})\, d\Omega + \frac{1}{\epsilon} \int_\Omega (\nabla \cdot \mathbf{u}_\epsilon)(\nabla \cdot \mathbf{v})\, d\Omega & \\ - \int_{\Gamma_g} \mathbf{t}_\epsilon \cdot \gamma_g \mathbf{v}\, d\Gamma = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega \quad \forall \mathbf{v} \in \mathbf{V}\,, & \\ \int_{\Gamma_g} \mathbf{s}^* \cdot \gamma_g \mathbf{u}_\epsilon\, d\Gamma = \int_{\Gamma_g} \mathbf{s}^* \cdot \mathbf{g} \quad \forall \mathbf{s}^* \in \mathbf{W}^*\,. & \end{aligned} \tag{4.20}$$

Once the saddle point $(\mathbf{u}_\epsilon, \mathbf{t}_\epsilon)$ of $\mathfrak{E}$ is sought, the pressure can be recovered from $p_\epsilon = -\dfrac{1}{\epsilon} \nabla \cdot \mathbf{u}_\epsilon$ and the Lagrange multiplier is given by $\mathbf{t}_\epsilon = (-p_\epsilon \mathbf{n} + 2\nu D(\mathbf{u}_\epsilon) \cdot \mathbf{n})$ on Γ_g. Hence the stress force plays the role of a Lagrange multiplier enforcing the inhomogeneous boundary condition.

We are now ready to show the existence and convergence of the solutions for the penalized system.

THEOREM 4.5. *Let* $\{(\lambda, (\mathbf{u}(\lambda), \lambda p(\lambda), \lambda \mathbf{t}(\lambda))) \mid \lambda = \dfrac{1}{\nu} \in \Lambda\}$ *be a branch of regular solutions of (4.9). Then, there exists a neighborhood* $\mathcal{O}$ *of the origin in* $\mathbf{V} \times S \times \mathbf{W}^*$ *and for* $\epsilon \le \epsilon_0$ *small enough, a unique* $\mathcal{C}^2$ *branch* $\{(\lambda, (\mathbf{u}_\epsilon(\lambda), \lambda p_\epsilon(\lambda), \lambda \mathbf{t}_\epsilon(\lambda))) \mid \lambda \in \Lambda\}$ *of the penalized system (1.1)–(1.3) such that* $\mathbf{u}_\epsilon(\lambda) - \mathbf{u}(\lambda) \in \mathcal{O}$ *for all* $\lambda \in \Lambda$. *Moreover, there exists a positive constant* C*, independent of* ϵ *and* λ*, such that*

$$\|\mathbf{u}_\epsilon(\lambda) - \mathbf{u}(\lambda)\|_{1,\Omega} + \|p_\epsilon(\lambda) - p(\lambda)\|_{0,\Omega} + \|\mathbf{t}_\epsilon(\lambda) - \mathbf{t}(\lambda)\|_{-1/2,\Gamma_g} \le C\epsilon \qquad \forall \lambda \in \Lambda\,. \tag{4.21}$$

Proof. Let $\boldsymbol{\psi} = (\mathbf{u}, p, \mathbf{t})$. We note that the first and second Fréchet derivatives of G with respect to $\boldsymbol{\psi}$ yield

$$D_{\boldsymbol{\psi}} G(\lambda, (\mathbf{u}, p, \mathbf{t})) \cdot (\mathbf{v}, q, \mathbf{s}^*) = \lambda \left(((\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}), \mathbf{0} \right) \in \mathbf{Y}$$

and

$$D_{\boldsymbol{\psi}\boldsymbol{\psi}} G(\lambda, (\mathbf{u}, p, \mathbf{t})) \cdot ((\mathbf{v}, q, \mathbf{s}^*), (\widehat{\mathbf{v}}, \widehat{q}, \widehat{\mathbf{s}}^*)) = \lambda \left(((\mathbf{v} \cdot \nabla)\widehat{\mathbf{v}} + (\widehat{\mathbf{v}} \cdot \nabla)\mathbf{v}), \mathbf{0} \right) \in \mathbf{Y}$$

for all $(\mathbf{v}, q, \mathbf{s}^*)$, $(\widehat{\mathbf{v}}, \widehat{q}, \widehat{\mathbf{s}}^*) \in \mathbf{X}$. It is clear that G belongs to $\mathcal{C}^2$ and that $D_{\boldsymbol{\psi}} G$ and $D_{\boldsymbol{\psi}\boldsymbol{\psi}} G$ are bounded on all bounded subset of $\Lambda \times \mathbf{X}$ by the Sobolev embedding theorem. It should be noted that $(\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}$ belongs to $\mathbf{L}^{3/2}(\Omega)$ and is compactly embedded in $\mathbf{V}^*$. Hence, the condition (4.3) is satisfied. Since $\mathbf{Z} = \mathbf{L}^{3/2}(\Omega) \times \{\mathbf{0}\}$ is compactly embedded in $\mathbf{Y} = \mathbf{V}^* \times \mathbf{W}$, the condition (4.5) follows directly from the condition (4.4). To verify the condition (4.4), we consider (4.7) and (4.11). By Theorem 4.4, (4.7) has a unique solution $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{t}})$ in $\mathbf{X} = \mathbf{V} \times S \times \mathbf{W}^*$. By subtracting (4.7) from (4.11), we obtain

$$\begin{aligned} a(\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \widehat{p}_\epsilon - \widehat{p}) - < \widehat{\mathbf{t}}_\epsilon - \widehat{\mathbf{t}}, \mathbf{v} >_{-1/2, \Gamma_g} = 0 \ \forall \mathbf{v} \in \mathbf{V}\,, \\ b(\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}, q) = \epsilon\, (\widehat{p}_\epsilon - \widehat{p}, q)_0 + \epsilon (\widehat{p}, q)_0 \quad \forall q \in S\,, \\ < \mathbf{s}, \widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}} >_{-1/2, \Gamma_g} = 0 \quad \forall \mathbf{s}^* \in \mathbf{W}^*\,. \end{aligned} \tag{4.22}$$

Taking $\mathbf{v}$ in $\mathbf{H}^1_0(\Omega)$, the first equation of (4.22) is reduced to

$$a(\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \widehat{p}_\epsilon - \widehat{p}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)\,. \tag{4.23}$$

Since $b(\cdot, \cdot)$ satisfy the div-stability condition, (4.23) yields

$$\frac{1}{C} \|\widehat{p}_\epsilon - \widehat{p}\|_0 \leq \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{H}^1_0(\Omega)} \frac{b(\mathbf{v}, \widehat{p}_\epsilon - \widehat{p})}{\|\mathbf{v}\|_1} \leq \|\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}\|_1\,,$$

for some positive constant C and hence

$$\|\widehat{p}_\epsilon - \widehat{p}\|_0 \leq C \|\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}\|_1\,. \tag{4.24}$$

By taking $\mathbf{v} = \widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}$ and $q = \widehat{p}_\epsilon - \widehat{p}$ in (4.22) and (4.23), and by substituting the second equation of (4.22) to (4.23), we derive

$$\begin{aligned} a(\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}, \widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}) &= -\epsilon\,(\widehat{p}_\epsilon - \widehat{p}, \widehat{p}_\epsilon - \widehat{p})_0 - \epsilon\,(\widehat{p}, \widehat{p}_\epsilon - \widehat{p})_0 \\ &\le -\epsilon\,(\widehat{p}, \widehat{p}_\epsilon - \widehat{p})_0\,. \end{aligned}$$

Therefore, combining (4.24) with Korn's Lemma, we obtain

$$\|\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}\|_1 \le C\epsilon\,\|\widehat{p}\|_0 \quad \text{and} \quad \|\widehat{p}_\epsilon - \widehat{p}\|_0 \le C^2\epsilon\,\|\widehat{p}\|_0\,.$$

Finally, the first equation of (4.22) yields

$$\begin{aligned} <\widehat{\mathbf{t}}_\epsilon - \widehat{\mathbf{t}}, \gamma_g \mathbf{v}>_{-1/2,\Gamma_g} &\le \|\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}\|_1 \|\mathbf{v}\|_1 + \|\nabla\cdot\mathbf{v}\|_0 \|\widehat{p}_\epsilon - \widehat{p}\|_0 \\ &\le (\|\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}\|_1 + \|\widehat{p}_\epsilon - \widehat{p}\|_0)\|\mathbf{v}\|_1 \quad \forall \mathbf{v} \in \mathbf{V}\,, \end{aligned}$$

whence from Lemma 2.1, we obtain

$$\|\widehat{\mathbf{t}}_\epsilon - \widehat{\mathbf{t}}\|_{-1/2,\Gamma_g} \le \|\widehat{\mathbf{u}}_\epsilon - \widehat{\mathbf{u}}\|_1 + \|\widehat{p}_\epsilon - \widehat{p}\|_0 \le (C + C^2)\epsilon\,\|\widehat{p}\|_0\,.$$

Therefore, we have shown that

$$\lim_{\epsilon\to 0} \|(T^\epsilon - T)(\widehat{\mathbf{f}}, \widehat{\mathbf{g}})\|_{\mathbf{X}} = 0 \quad \text{for all} \quad (\widehat{\mathbf{f}}, \widehat{\mathbf{g}}) \in \mathbf{V}^* \times \mathbf{W}\,.$$

Hence, Theorem 4.5 immediately follows from Theorem 4.1. □ □

Theorem 4.5 implies that regular solutions of the penalized system converge to that of the primal system. Physically, the associated error amounts to net fluid loss or gain caused by the penalization. Since the solutions of Navier–Stokes equations are regular for almost all Reynolds numbers, the solutions of the penalized Navier–Stokes system are locally unique.

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