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A STUDY ON THE SCHUR ALGEBRA OF SIZE 4

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Abstract. In this paper, we will show that the minimal number of generators of any four dimensional, faithful, $\mathcal{B}$(Schur algebra of size 4)-module is two. This result can be applied to classify the isomorphism classes of the class $\{\mathcal{B}\ltimes N^2 \mid N$ is a faithful, $\mathcal{B}$-module with $dim_k(N) = 4\}$.

1. Introduction

In this paper, k will denote an arbitrary field. Throughout this paper, we will denote the Schur algebra of size 4 by $\mathcal{B}$. Thus,

$$\mathcal{B} = \left\{ \begin{pmatrix} x & 0 & a & b \\ 0 & x & c & d \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \mid x, a, b, c, d \in k \right\}.$$

Recall that a commutative k-algebra R is a (B, N)-construction if R is k-algebra isomorphic to $B \ltimes N^\ell$, the idealization of a B-module N, for some finite dimensional, commutative, local, k-algebra B and finitely generated, faithful, B-module N and natural number ℓ.

In [1], W.C.Brown and F.W.Call showed that the Courter's algebra $\mathcal{C}$ is a (B, N)-construction, where B is the Schur algebra of size 4, $N = k^4$, and $\ell = 2$. That is, $\mathcal{C} \cong \mathcal{B}\ltimes(k^4)^2$. But, as we will see in the next section, there are at least two nonisomorphic $\mathcal{B}$-modules. Thus, it is very natural to be asked how many isomorphism classes can be constructed by varying the faithful, $\mathcal{B}$-module N.

Let $M\mathcal{B}(4) = \{N \mid N$ is a faithful, $\mathcal{B}$-module and $dim_k(N) = 4\}$. Then, we will show the minimal number of generators of N in $M\mathcal{B}(4)$

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is two. This can be a fundamental building block to classify the isomorphism classes of the class $\{R \mid R \text{ is a } k\text{-algebra and } R \cong \mathcal{B} \ltimes N^2 \text{ for some } N \in M\mathcal{B}(4)\ \}$.

2. Classification of $M\mathcal{B}(4)$

We will first show the set $M\mathcal{B}(4)$ has at least two isomorphism classes. To see this, we first need a $\mathcal{B}$-module presentation of k^4. We will denote the i, j-th matrix unit of $M_{4\times 4}(k)$ by E_{ij}. Notice that $E_{ij} \in \mathcal{B}$ if $i = 1, 2, j = 3, 4$.

LEMMA 2.1. *Let*

$$(1) \qquad A = \begin{pmatrix} E_{23} & E_{24} & E_{13} & E_{14} & O & O \\ -E_{13} & -E_{14} & O & O & E_{23} & E_{24} \end{pmatrix} \in M_{2\times 6}(\mathcal{B}).$$

Then, $\mathcal{B}^2/CS(A) \in M\mathcal{B}(4)$.

Proof. Obviously, $\mathcal{B}^2/CS(A)$ is a finitely generated, $\mathcal{B}$-module. Since $dim_k(\mathcal{B}^2) = 10$ and $dim_k(CS(A)) = 6, dim_k(\mathcal{B}^2/CS(A)) = 4$. Suppose $r \in Ann_{\mathcal{B}}(\mathcal{B}^2/CS(A))$. Then, $r\begin{pmatrix} I_4 \\ O \end{pmatrix}, r\begin{pmatrix} O \\ I_4 \end{pmatrix} \in CS(A)$. Thus, $\begin{pmatrix} r \\ O \end{pmatrix}, \begin{pmatrix} O \\ r \end{pmatrix} \in CS(A)$ which implies that for some $x_i, y_j \in \mathcal{B}$, $1 \le i, j \le 6$

$$(2) \qquad \begin{aligned} r &= x_1E_{23} + x_2E_{24} + x_3E_{13} + x_4E_{14} \\ 0 &= -x_1E_{13} - x_2E_{14} + x_5E_{23} + x_6E_{24} \\ 0 &= y_1E_{23} + y_2E_{24} + y_3E_{13} + y_4E_{14} \\ r &= -y_1E_{13} - y_2E_{14} + y_5E_{23} + y_6E_{24} \end{aligned}$$

Since $J(\mathcal{B})^2 = (0)$, we can assume $x_i, y_j \in k = kI_4$ for $1 \le i, j \le 6$. The second and third equations in (2) imply $x_1, x_2, x_5, x_6, y_1, y_2, y_3, y_4$ are all zero. Thus, $r = x_3E_{13} + x_4E_{14} = y_5E_{23} + y_6E_{24}$. Therefore, $r = 0$. Hence, $Ann_{\mathcal{B}}(\mathcal{B}^2/CS(A)) = (0)$ and $\mathcal{B}^2/CS(A)$ is a faithful, $\mathcal{B}$-module. □ □

LEMMA 2.2. *Let A be the matrix in Equation (1). Then $\mathcal{B}^2/CS(A)$ is $\mathcal{B}$-module isomorphic to k^4.*

Proof. Let $f : \mathcal{B}^2 \longrightarrow k^4$ be the map defined by $f\begin{pmatrix} x \\ y \end{pmatrix} = \varepsilon_2 x + \varepsilon_1 y$. Here, $\varepsilon_1 = (1, 0, 0, 0)$ and $\varepsilon_2 = (0, 1, 0, 0)$. Then, f is a surjective, $\mathcal{B}$-module homomorphism. If $\begin{pmatrix} z \\ w \end{pmatrix} \in ker\ f$, then $z = a_1 I_4 + a_2 E_{13} + a_3 E_{14} + a_4 E_{23} + a_5 E_{24}$ and $w = b_1 I_4 + b_2 E_{13} + b_3 E_{14} + b_4 E_{23} + b_5 E_{24}$ for some $a_i, b_i \in k, i = 1, \ldots, 5$. Since $f\begin{pmatrix} z \\ w \end{pmatrix} = \varepsilon_2 z + \varepsilon_1 w = 0, a_1 = b_1 = 0, b_2 = -a_4$, and $b_3 = -a_5$.

Thus,

$$\begin{aligned}\begin{pmatrix} z \\ w \end{pmatrix} = a_2 \begin{pmatrix} E_{13} \\ O \end{pmatrix} + a_3 \begin{pmatrix} E_{14} \\ O \end{pmatrix} + a_4 \begin{pmatrix} E_{23} \\ -E_{13} \end{pmatrix} \\ + a_5 \begin{pmatrix} E_{24} \\ -E_{14} \end{pmatrix} + b_4 \begin{pmatrix} O \\ E_{23} \end{pmatrix} + b_5 \begin{pmatrix} O \\ E_{24} \end{pmatrix}\end{aligned}.$$

Hence, $\begin{pmatrix} z \\ w \end{pmatrix} \in CS(A)$. It is easy to check that $CS(A) \subseteq kerf$. Therefore, $CS(A) = kerf$. Hence, $\mathcal{B}^2/CS(A) \cong k^4$ as $\mathcal{B}$-modules. □ □

We can now construct a faithful, $\mathcal{B}$-module of dimension 4 which is not isomorphic to k^4 as $\mathcal{B}$-modules.

THEOREM 2.3:. *Let*

$$C = \begin{pmatrix} E_{13} & E_{14} & E_{23} & E_{24} & O & O \\ E_{24} & E_{23} & O & O & E_{13} & E_{14} \end{pmatrix} \in M_{2\times 6}(\mathcal{B}). \tag{3}$$

Then, $\mathcal{B}^2/CS(C) \in M\mathcal{B}(4)$ and $\mathcal{B}^2/CS(C)$ is not $\mathcal{B}$-module isomorphic to k^4.

Proof. Obviously, $\mathcal{B}^2/CS(C)$ is a finitely generated, $\mathcal{B}$-module. Since $dim_k(\mathcal{B}^2) = 10$ and $dim_k(CS(C)) = 6$, $dim_k(\mathcal{B}^2/CS(C)) = 4$. Suppose $r \in Ann_{\mathcal{B}}(\mathcal{B}^2/CS(C))$. Then, $\begin{pmatrix} r \\ O \end{pmatrix}, \begin{pmatrix} O \\ r \end{pmatrix} \in CS(C)$ which implies that for some $x_i, y_j \in \mathcal{B}$, $1 \leq i, j \leq 6$

$$
\begin{aligned}
r &= x_1E_{13} + x_2E_{14} + x_3E_{23} + x_4E_{24} \\
0 &= x_1E_{24} + x_2E_{23} + x_5E_{13} + x_6E_{14} \\
0 &= y_1E_{13} + y_2E_{14} + y_3E_{23} + y_4E_{24} \\
r &= y_1E_{24} + y_2E_{23} + y_5E_{13} + y_6E_{14}
\end{aligned} \tag{4}
$$

Since $J(\mathcal{B})^2 = (0)$, we can assume $x_i, y_j \in k = kI_4$ for $1 \leq i, j \leq 6$. The second and third equations in (4) imply $x_1, x_2, x_5, x_6, y_1, y_2, y_3, y_4$ are all zero. Thus, $r = x_3E_{23} + x_4E_{24} = y_5E_{13} + y_6E_{14}$. Therefore, $r = 0$. Hence, $Ann_{\mathcal{B}}(\mathcal{B}^2/CS(C)) = (0)$ and $\mathcal{B}^2/CS(C) \in M\mathcal{B}(4)$.

Suppose $\mathcal{B}^2/CS(C)$ is $\mathcal{B}$-module isomorphic to k^4. Then, there exists a $\mathcal{B}$-module isomorphism $g : \mathcal{B}^2/CS(C) \longrightarrow k^4$. Let $\beta_1 = \begin{pmatrix} I_4 \\ O \end{pmatrix}^- = \begin{pmatrix} I_4 \\ O \end{pmatrix} + CS(C) \in \mathcal{B}^2/CS(C)$. and $\beta_2 = \begin{pmatrix} O \\ I_4 \end{pmatrix}^-$. Then, $\mathcal{B}^2/CS(C) = \beta_1\mathcal{B}+\beta_2\mathcal{B}$. Since $k^4 = \varepsilon_1\mathcal{B} + \varepsilon_2\mathcal{B}, g(\beta_1) = \varepsilon_1x_1 + \varepsilon_2y_1$ and $g(\beta_2) = \varepsilon_1x_2 + \varepsilon_2y_2$ for some $x_i, y_i \in \mathcal{B}$, $i = 1, 2$. Notice that x_1 or y_1 is unit. To see this, suppose $x_1, y_1 \in J(\mathcal{B})$. Then, $g(\beta_1) = \varepsilon_1x_1 + \varepsilon_2y_1 \in k^4J(\mathcal{B})$. The inclusions

$$k^4 = g(\beta_1)\mathcal{B} + g(\beta_2)\mathcal{B} \subseteq k^4J(\mathcal{B}) + g(\beta_2)J(\mathcal{B}) \subseteq k^4$$

imply that $k^4 = k^4J(\mathcal{B}) + g(\beta_2)J(\mathcal{B})$. By Nakayama's Lemma, $k^4 = g(\beta_2)J(\mathcal{B})$. This implies $\mathcal{B}$ is isomorphic to k^4 as $\mathcal{B}$-modules and hence $dim_k(\mathcal{B})$= 4. Since $dim_k(\mathcal{B})$= 5, this is impossible. Hence, x_1 or y_1 is unit in $\mathcal{B}$. Similarly, x_2 or y_2 is unit.

Let A be the matrix given in Equation (1) and let f be the $\mathcal{B}$-module homomorphism given in the proof of Lemma 2.2. If $\begin{pmatrix} z \\ w \end{pmatrix} \in CS(C)$, then

$$
\begin{aligned}
f\begin{pmatrix} y_1z + y_2w \\ x_1z + x_2w \end{pmatrix} &= \varepsilon_1(x_1z + x_2w) + \varepsilon_2(y_1z + y_2w) \\
&= (\varepsilon_1x_1 + \varepsilon_2y_1)z + (\varepsilon_1x_2 + \varepsilon_2y_2)w \\
&= g(\beta_1)z + g(\beta_2)w \\
&= g(\beta_1z + \beta_2w) \\
&= g(0) = 0.
\end{aligned}
$$

Thus,

$$(5)\qquad \begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} y_1 z + y_2 w \\ x_1 z + x_2 w \end{pmatrix} \in kerf = CS(A).$$

Now, there are two cases to consider.

Case 1: Suppose x_1 is a unit. Since $\begin{pmatrix} E_{13} \\ E_{24} \end{pmatrix} \in CS(C)$, we have $\begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} E_{13} \\ E_{24} \end{pmatrix} \in CS(A)$ by the Equation (5). Hence,

$$\begin{aligned} \begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} E_{13} \\ E_{24} \end{pmatrix} &= a_1 \begin{pmatrix} E_{23} \\ -E_{13} \end{pmatrix} + a_2 \begin{pmatrix} E_{24} \\ -E_{14} \end{pmatrix} + a_3 \begin{pmatrix} E_{13} \\ O \end{pmatrix} \\ &= a_4 \begin{pmatrix} E_{14} \\ O \end{pmatrix} + a_5 \begin{pmatrix} O \\ E_{23} \end{pmatrix} + a_6 \begin{pmatrix} O \\ E_{24} \end{pmatrix}. \end{aligned}$$

for some $a_i \in k, 1 \leq i \leq 6$ (See the comments after Equation (2)). Thus,

$$(6)\qquad \begin{aligned} y_1 E_{13} + y_2 E_{24} &= a_1 E_{23} + a_2 E_{24} + a_3 E_{13} + a_4 E_{14} \\ x_1 E_{13} + x_2 E_{24} &= -a_1 E_{13} - a_2 E_{14} + a_5 E_{23} + a_6 E_{24}. \end{aligned}$$

Let $x_1 = t_1 I_4 + s_1$ with $t_1 \in k$ and $s_1 \in J(\mathcal{B})$. The first equation in (6) then implies $a_1 = a_4 = 0$. The second equation in (6) then implies $t_1 = 0$. Thus, $x_1 \in J(\mathcal{B})$. Since we are assuming x_1 is a unit, this is impossible.

Case 2: Suppose y_1 is a unit. Since $\begin{pmatrix} E_{23} \\ O \end{pmatrix} \in CS(C)$, we have $\begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} E_{23} \\ O \end{pmatrix} \in CS(A)$ by the Equation (5). Hence,

$$\begin{aligned} \begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} E_{23} \\ O \end{pmatrix} &= b_1 \begin{pmatrix} E_{23} \\ -E_{13} \end{pmatrix} + b_2 \begin{pmatrix} E_{24} \\ -E_{14} \end{pmatrix} + b_3 \begin{pmatrix} E_{13} \\ O \end{pmatrix} \\ &+ b_4 \begin{pmatrix} E_{14} \\ O \end{pmatrix} + b_5 \begin{pmatrix} O \\ E_{23} \end{pmatrix} + b_6 \begin{pmatrix} O \\ E_{24} \end{pmatrix}. \end{aligned}$$

for some $b_i \in k, 1 \leq i \leq 6$. Thus,

$$(7)\qquad \begin{aligned} y_1 E_{23} &= b_1 E_{23} + b_2 E_{24} + b_3 E_{13} + b_4 E_{14} \\ x_1 E_{23} &= -b_1 E_{13} - b_2 E_{14} + b_5 E_{23} + b_6 E_{24}. \end{aligned}$$

The second equation in (7) implies $b_1 = 0$ and the first equation in (7) implies $y_1 \in J(\mathcal{B})$. This is impossible. We conclude there is no $\mathcal{B}$-module isomorphism g between $\mathcal{B}^2/CS(C)$ and k^4. □ □

Thus, $M\mathcal{B}(4)$ has at least two isomorphism classes $[\mathcal{B}^2/CS(A)]$ and $[\mathcal{B}^2/CS(C)]$. We will denote the minimal number of generators of $\mathcal{B}$-module N by $\mu_{\mathcal{B}}(N)$.

THEOREM 2.4. *Let $N \in M\mathcal{B}(4)$. Then, $\mu_{\mathcal{B}}(N) = 2$.*

Proof. Since $dim_k(N) = 4$, $1 \leq \mu_{\mathcal{B}}(N) \leq 4$. Suppose $\mu_{\mathcal{B}}(N) = 1$. Then, $N = \alpha\mathcal{B}$ for some $\alpha \in N$. Let $f : \mathcal{B} \longrightarrow N$ be a map defined by $f(b) = \alpha b$ for $b \in \mathcal{B}$. Then, f is a $\mathcal{B}$-module epimorphism. If $b \in ker f$, then $\alpha b = 0$. Thus, $b \in Ann_{\mathcal{B}}(\alpha) = Ann_{\mathcal{B}}(\alpha\mathcal{B})$. Since N is a faithful, $\mathcal{B}$-module, $Ann_{\mathcal{B}}(\alpha\mathcal{B}) = (0)$. Therefore, $b = 0$ and hence f is a $\mathcal{B}$-module isomorphism. Thus, $5 = dim_k(\mathcal{B}) = dim_k(\alpha\mathcal{B}) = 4$. This is impossible. Hence, $2 \leq \mu_{\mathcal{B}}(N) \leq 4$.

Suppose $\mu_{\mathcal{B}}(N) = 4$. By Nakayama's Lemma, we have $\mu_{\mathcal{B}}(N) = dim_k(N/NJ(\mathcal{B}))$. Therefore, $dim_k(NJ(\mathcal{B})) = 0$. Thus, $NJ(\mathcal{B}) = (0)$. Since N is a faithful, $\mathcal{B}$-module, we conclude $J(\mathcal{B}) = (0)$. This is impossible.

Suppose $\mu_{\mathcal{B}}(N) = 3$. Then, $N = \alpha_1\mathcal{B}+\alpha_2\mathcal{B} + \alpha_3\mathcal{B}$ for some $\alpha_i, i = 1, 2, 3$. After relabeling the α_i's if need be, we can assume $\alpha_1, \alpha_2, \alpha_3$ satisfy precisely one of the following four conditions :

Case 1: $\alpha_i J(\mathcal{B}) = (0)$ for $i = 1, 2, 3$.
Case 2: $\alpha_i J(\mathcal{B}) = (0)$ for $i = 1, 2$ and $\alpha_3 J(\mathcal{B}) \neq (0)$.
Case 3: $\alpha_1 J(\mathcal{B}) = (0)$ and $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 2, 3$.
Case 4: $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 1, 2, 3$.

We will show all four cases lead to a contradiction.

Case 1: Suppose $\alpha_i J(\mathcal{B}) = (0)$ for all $i = 1, 2, 3$. Then, $NJ(\mathcal{B}) = (0)$. Since N is a faithful, $\mathcal{B}$-module, $J(\mathcal{B}) = (0)$. This is impossible.

Case 2: Suppose $\alpha_i J(\mathcal{B}) = (0)$ for all $i = 1, 2$ and $\alpha_3 J(\mathcal{B}) \neq (0)$. Suppose $\alpha_3 b = 0$ for some $b \in \mathcal{B}$. If b is a unit, then $\alpha_3 = 0$. This is impossible. Thus, $b \in J(\mathcal{B})$. Hence, $b \in Ann_{\mathcal{B}}(N)$. Since N is a faithful, $\mathcal{B}$-module, we conclude $b = 0$. Thus, $Ann_{\mathcal{B}}(\alpha_3) = (0)$ and hence $\mathcal{B} \cong \alpha_3\mathcal{B} \subseteq N$ as $\mathcal{B}$-modules. Since $dim_k(\mathcal{B}) = 5$, this is impossible.

Case 3: Suppose $\alpha_1 J(\mathcal{B}) = (0)$ and $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 2, 3$. Since $\left\{ \beta_1 = \begin{pmatrix} I_4 \\ O \\ O \end{pmatrix}, \beta_2 = \begin{pmatrix} O \\ I_4 \\ O \end{pmatrix}, \beta_3 = \begin{pmatrix} O \\ O \\ I_4 \end{pmatrix} \right\}$ is a free $\mathcal{B}$-module basis of $\mathcal{B}^3$, the map $\varphi : \mathcal{B}^3 \longrightarrow N$ defined by $\varphi(\sum_{i=1}^{3} \beta_i b_i) = \sum_{i=1}^{3} \alpha_i b_i$, $b_i \in \mathcal{B}$, $i = 1, 2, 3$ is a well defined $\mathcal{B}$-module epimorphism. Thus, $\mathcal{B}^3 / ker\varphi \cong N$ as $\mathcal{B}$-modules. Since $dim_k(\mathcal{B}^3) = 15$ and $dim_k(N) = 4$, $dim_k(ker\varphi) = 11$. Hence, $ker\varphi$ has the following form

$$ker\varphi = \sum_{i=1}^{11} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \mathcal{B}, \quad x_i, y_i, z_i \in \mathcal{B}, i = 1, \ldots, 11.$$

Furthermore, if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in ker\varphi$, then x, y, z are not units in $\mathcal{B}$. For example, suppose x is a unit in $\mathcal{B}$. Since $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in ker\varphi$, $\alpha_1 = (-1/x)(\alpha_2 y + \alpha_3 z)$. Thus, $\mu_{\mathcal{B}}(N) < 3$ which is impossible.

Since $J(\mathcal{B})^2 = (0), ker\varphi$ can be written in the following form

$$ker\varphi = \oplus_{i=1}^{11} k \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}.$$

Here, $x_i, y_i, z_i \in J(\mathcal{B}), i = 1, \ldots, 11$. Since $\alpha_1 J(\mathcal{B}) = (0), (\beta_1 + ker\varphi)J(\mathcal{B}) = (0)$ in $\mathcal{B}^3/ker\varphi$. Thus, $\begin{pmatrix} J(\mathcal{B}) \\ O \\ O \end{pmatrix} \subseteqq ker\varphi$. Since $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 2, 3$, $\quad 1 \leq dim_k(Ann_{\mathcal{B}}(\alpha_i)) < 4$ for $i = 2, 3$. Therefore, we have the following six subcases to consider.

Subcase 1: $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 1$ for $i = 2, 3$
Subcase 2: $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 2$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$
Subcase 3: $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 2$ for $i = 2, 3$
Subcase 4: $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$
Subcase 5: $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 2$

Subcase 6: $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 3$ for $i = 2, 3$

We will show all six subcases lead to a contradiction.

Subcase 1: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 1$ for $i = 2, 3$. Let $Ann_{\mathcal{B}}\ (\alpha_i) = ks_i, s_i \in J(\mathcal{B}),\ i = 2, 3$. Then, $\begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix} \in ker\varphi$.

Since $\alpha_1 J(\mathcal{B}) = (0)$, $\begin{pmatrix} J(\mathcal{B}) \\ O \\ O \end{pmatrix} \subseteq ker\varphi$. Let

$$\left\{ \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \right\}$$

be a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$ and $x_i \in J(\mathcal{B})$ for $i = 1, \ldots, 5, x_i \in L(E_{13}, E_{14}, E_{23}, E_{24})$ for $i = 1, \ldots, 5$. Thus,

$$\left\{ \delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \delta_5 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix}, \delta_9 = \begin{pmatrix} O \\ y_3 \\ z_3 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ y_4 \\ z_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_5 \\ z_5 \end{pmatrix} \right\}$$

is a basis of $ker\varphi$. Therefore, $ker\varphi$ can be written in the following form

$$ker\varphi = \begin{pmatrix} J \\ O \\ O \end{pmatrix} \oplus k \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix} \oplus k \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix} \oplus \sum_{i=1}^{5} k \begin{pmatrix} O \\ y_i \\ z_i \end{pmatrix}.$$

Since $dim_k(J(\mathcal{B})) = 4, \{s_2, y_1, \ldots, y_5\}$ is a linearly dependent set. Thus, there exist $d, c_1, \ldots, c_5 \in k$ not all zero such that $ds_2 + c_1y_1 + \cdots + c_5y_5 = 0$. If $c_i = 0$ for all $i = 1, \ldots, 5$, then $d \neq 0$ and $ds_2 = 0$. This implies $s_2 = 0$. This is impossible since $\begin{pmatrix} 0 \\ s_2 \\ 0 \end{pmatrix}$ is a basis vector of $ker\varphi$. Hence, some c_i is not zero. We can assume $c_5 \neq 0$. Thus, $y_5 \in L(s_2, y_1, \ldots, y_4)$. We can repeat this proof on $s_2, y_1, \ldots, y_4$ and assume $y_4 \in L(s_2, y_1, y_2, y_3)$. Hence, we may assume $y_4, y_5 \in L(s_2, y_1, y_2, y_3)$. Therefore, $y_4 = ds_2 + c_1y_1 + c_2y_2 + c_3y_3$ for some $d, c_1, c_2, c_3 \in k$. If $d\delta_5 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 - \delta_{10} = 0$, then $\{\delta_5, \delta_7, \delta_8, \delta_9, \delta_{10}\}$ is linearly dependent which is impossible. Thus, $d\delta_5 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 - \delta_{10} = \begin{pmatrix} O \\ O \\ z \end{pmatrix}$ with $z \neq 0$ in $J(\mathcal{B})$. If $z = ts_3$ for some $t \in k$, then $d\delta_5 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 - \delta_{10} - t\delta_6 = 0$ and $\{\delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}\}$ is linearly dependent which is impossible. Thus, $\begin{pmatrix} O \\ O \\ z \end{pmatrix} \in ker\varphi \backslash k\delta_6$. Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_3)) \geq 2$. This is a contradiction.

Subcase 2: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 2$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$. Then, $Ann_{\mathcal{B}}(\alpha_2) = ks_1 + ks_2$ and $Ann_{\mathcal{B}}(\alpha_3) = ks_3$ for some $s_i \in J(\mathcal{B}), i = 1, 2, 3$. Let

$$\left\{ \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \right\}$$

be a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$ and $x_i \in J(\mathcal{B})$ for $i =$

$1, \ldots, 4$, $x_i \in L(E_{13}, E_{14}, E_{23}, E_{24})$ for $i = 1, \ldots, 4$. Thus,

$$\left\{ \delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \right.$$
$$\delta_5 = \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix},$$
$$\left. \delta_9 = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ y_3 \\ z_3 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_4 \\ z_4 \end{pmatrix} \right\}$$

is a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_3, z_1, \ldots, z_4\}$ is a linearly dependent set. Thus, there exist $d, c_1, \ldots, c_4 \in k$ not all zero such that $ds_3 + c_1z_1 + \cdots + c_4z_4 = 0$. If $c_i = 0$ for all $i = 1, \ldots, 4$, then $d \neq 0$ and $ds_3 = 0$. This implies $s_3 = 0$. This is impossible since $\begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}$ is a basis vector of $ker\varphi$. Hence, some c_i is not zero. We can assume $c_4 \neq 0$. Thus, $z_4 = ds_3 + c_1z_1 + c_2z_2 + c_3z_3$ for some $d, c_1, c_2, c_3 \in k$. If $d\delta_7 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} = 0$, then $\{\delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $d\delta_7 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ y \\ O \end{pmatrix}$ with $y \neq 0$ in $J(\mathcal{B})$. If $y = t_1s_1 + t_2s_2$ for some $t_1, t_2 \in k$, then $d\delta_7 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} - t_1\delta_5 - t_2\delta_6 = 0$ and $\{\delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $\begin{pmatrix} O \\ y \\ O \end{pmatrix} \in ker\varphi \backslash k\delta_5 + k\delta_6$. Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_2)) \geq 3$. This is a contradiction.

Subcase 3: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 2$ for $i = 2, 3$. Then, $Ann_{\mathcal{B}}(\alpha_2) = ks_1 + ks_2$ and $Ann_{\mathcal{B}}(\alpha_3) = ks_3 + ks_4$ for some $s_i \in$

$J(\mathcal{B}), i = 1, 2, 3, 4$. Let

$$\left\{ \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right\}$$

be a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$ and $x_i \in J(\mathcal{B})$ for $i = 1, 2, 3$, $x_i \in L(E_{13}, E_{14}, E_{23}, E_{24})$ for $i = 1, 2, 3$. Thus,

$$\left\{ \delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \delta_5 = \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \delta_9 = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_3 \\ z_3 \end{pmatrix} \right\}$$

is a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4, \{s_1, s_2, y_1, y_2, y_3\}$ is a linearly dependent set. Thus, there exist $d_1, d_2, c_1, c_2, c_3 \in k$ not all zero such that $d_1 s_1 + d_2 s_2 + c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$. If $c_i = 0$ for all $i = 1, 2, 3$, then $d_1 s_1 + d_2 s_2 = 0$. Since s_1, s_2 are linearly independent vectors in $J(\mathcal{B}), d_1 = d_2 = 0$. This is impossible. Thus, $c_i \neq 0$ for some $1 \leq i \leq 3$. We can assume $c_3 \neq 0$. Hence, $y_3 = d_1 s_1 + d_2 s_2 + c_1 y_1 + c_2 y_2$ for some $d_1, d_2, c_1, c_2 \in k$. If $d_1\delta_5 + d_2\delta_6 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = 0$, then $\{\delta_5, \delta_6, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $d_1\delta_5 + d_2\delta_6 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ z \end{pmatrix}$ with $z \neq 0$ in $J(\mathcal{B})$. If $z = t_3 s_3 + t_4 s_4$ for some $t_3, t_4 \in k$, then $d_1\delta_5 + d_2\delta_6 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} - t_3\delta_7 - t_4\delta_8 = 0$. This is a contradiction. Thus, $\begin{pmatrix} O \\ O \\ z \end{pmatrix} \in ker\varphi \backslash k\delta_7 + k\delta_8$.

Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_3)) \geq 3$ and this is a contradiction.

Subcase 4: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$. Then, $Ann_{\mathcal{B}}(\alpha_2) = ks_1 + ks_2 + ks_3$ and $Ann_{\mathcal{B}}(\alpha_3) = ks_4$ for some $s_i \in J(\mathcal{B}), i = 1, 2, 3, 4$. Let

$$\left\{ \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right\}$$

be a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$ and $x_i \in J(\mathcal{B}), x_i \in L(E_{13}, E_{14}, E_{23}, E_{24})$ for $i = 1, 2, 3$. Thus,

$$\left\{ \delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \delta_5 = \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \delta_9 = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix} \delta_{10} = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_3 \\ z_3 \end{pmatrix} \right\}$$

is a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4, \{s_1, s_2, s_3, y_1, y_2, y_3\}$ is a linearly dependent set. Thus, there exist $d_1, d_2, d_3, c_1, c_2, c_3 \in k$ not all zero such that $d_1s_1 + d_2s_2 + d_3s_3 + c_1y_1 + c_2y_2 + c_3y_3 = 0$. If $c_i = 0$ for all $i = 1, 2, 3$, then $d_1s_1 + d_2s_2 + d_3s_3 = 0$ Since s_1, s_2, s_3 are linearly independent vectors in $J(\mathcal{B}), d_1 = d_2 = d_3 = 0$. This is impossible. Thus, $c_i \neq 0$ for some i. We can assume $c_3 \neq 0$. Hence, $y_3 = d_1s_1 + d_2s_2 + d_3s_3 + c_1y_1 + c_2y_2$ for some $d_1, d_2, d_3, c_1, c_2 \in k$. If $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = 0$, then $\{\delta_5, \delta_6, \delta_7, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ z \end{pmatrix}$ with $z \neq 0$ in $J(\mathcal{B})$. If $z = ts_4$ for some $t \in k$, then $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} - t\delta_8 = 0$. This is impossible since $\{\delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent. Thus, $\begin{pmatrix} O \\ O \\ z \end{pmatrix} \in$ $ker\varphi \backslash k\delta_8$. Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_3)) \geq 2$ and this is a contradiction.

Subcase 5: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 2$. Then, $Ann_{\mathcal{B}}(\alpha_2) = ks_1 + ks_2 + ks_3$ and $Ann_{\mathcal{B}}(\alpha_3) = ks_4 + ks_5$ for some $s_i \in J(\mathcal{B})$, $i = 1, 2, 3, 4, 5$. Let

$$\left\{ \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\}$$

be a basis of $ker\varphi$. Since $J(\mathcal{B}) = L(E_{13}, E_{14}, E_{23}, E_{24})$ and $x_1, x_2 \in J(\mathcal{B}), x_1, x_2 \in L(E_{13}, E_{14}, E_{23}, E_{24})$. Thus,

$$\left\{ \delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \right.$$
$$\delta_5 = \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix},$$
$$\left. \delta_9 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix} \right\}$$

is a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4, \{s_1, s_2, s_3, y_1, y_2\}$ is a linearly dependent set. Thus, there exist $d_1, d_2, d_3, c_1, c_2 \in k$ not all zero such that $d_1s_1 + d_2s_2 + d_3s_3 + c_1y_1 + c_2y_2 = 0$. If $c_1 = c_2 = 0$, then $d_1s_1 + d_2s_2 + d_3s_3 = 0$. Since s_1, s_2, s_3 are linearly independent vectors in $J(\mathcal{B}), d_1 = d_2 = d_3 = 0$. This is impossible. Thus, $c_i \neq 0$ for some i. We can assume $c_2 \neq 0$. Hence, $y_2 = d_1s_1 + d_2s_2 + d_3s_3 + c_1y_1$ for some $d_1, d_2, d_3, c_1 \in k$. If $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_{10} - \delta_{11} = 0$, then $\{\delta_5, \delta_6, \delta_7, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus,

$d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ z \end{pmatrix}$ with $z \neq 0$ in $J(\mathcal{B})$. If $z = t_4 s_4 + t_5 s_5$ for some $t_4, t_5 \in k$, then $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_{10} - \delta_{11} - t_4\delta_8 - t_5\delta_9 = 0$. This is again impossible since $\{\delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent. Thus, $\begin{pmatrix} O \\ O \\ z \end{pmatrix} \in ker\varphi \backslash k\delta_8 + k\delta_9$. Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_3)) \geq 3$ which is a contradiction.

Subcase 6: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 3$ for $i = 2, 3$. Note that

$$\begin{aligned} &dim_k(Ann_{\mathcal{B}}(\alpha_2)) + dim_k(Ann_{\mathcal{B}}(\alpha_3)) = \\ &dim_k(Ann_{\mathcal{B}}(\alpha_2) + Ann_{\mathcal{B}}(\alpha_3)) + dim_k(Ann_{\mathcal{B}}(\alpha_2) \cap Ann_{\mathcal{B}}(\alpha_3)). \end{aligned} \tag{8}$$

Since $dim_k(Ann_{\mathcal{B}}(\alpha_2) + Ann_{\mathcal{B}}(\alpha_3)) \leq dim_k(J(\mathcal{B})) = 4$, Equation (8) implies $dim_k(Ann_{\mathcal{B}}(\alpha_2) \cap Ann_{\mathcal{B}}(\alpha_3)) \geq 2$ Thus, there is $0 \neq b \in Ann_{\mathcal{B}}(\alpha_2) \cap Ann_{\mathcal{B}}(\alpha_3)$. This is a contradiction. We have now shown any of the subcases in Case 3 lead to a contradiction. Hence, Case 3 is impossible.

Case 4: Suppose $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 1, 2, 3$. Let $n_i = dim_k(Ann_{\mathcal{B}}(\alpha_i))$. By relabeling the $\alpha_i's$ if need be, there are ten subcases to consider.

Subcase 1: Suppose $n_i = 1$ for $i = 1, 2, 3$.
Subcase 2: Suppose $n_1 = 2, n_2 = n_3 = 1$.
Subcase 3: Suppose $n_1 = n_2 = 2, n_3 = 1$.
Subcase 4: Suppose $n_i = 2$ for $i = 1, 2, 3$.
Subcase 5: Suppose $n_1 = 3, n_2 = n_3 = 1$.
Subcase 6: Suppose $n_1 = 3, n_2 = 2, n_3 = 1$.
Subcase 7: Suppose $n_1 = 3, n_2 = n_3 = 2$.
Subcase 8: Suppose $n_1 = n_2 = 3, n_3 = 1$.
Subcase 9: Suppose $n_1 = n_2 = 3, n_3 = 2$.
Subcase 10: Suppose $n_i = 3$ for $i = 1, 2, 3$.

A proof similar to that given in Case 3 will show that Subcase 1 through Subcase 9 are impossible. Subcase 10 is also impossible. To see this, let V be a vector space and suppose $W_i, i = 1, 2, 3$ are subspaces of V. Suppose $dim_k(V) = n$. Then, we have the following equation.

(9)

$$dim_k(W_1 \cap W_2 \cap W_3) = n - \sum_{i=1}^{3}(n - dim_k(W_i)) + \{(n - dim_k(W_1 + W_2)) + (n - dim_k((W_1 \cap W_2) + W_3))\}.$$

Suppose $V = \mathcal{B}$ and $W_i = Ann_{\mathcal{B}}(\alpha_i), i = 1, 2, 3$. Then, Equation (9) implies $dim_k(W_1 \cap W_2 \cap W_3) = 9 - dim_k(W_1 + W_2) - dim_k((W_1 \cap W_2) + W_3)$. Since $dim_k(W_1 + W_2) \leq 4$ and $dim_k((W_1 \cap W_2) + W_3) \leq 4$, we have $dim_k(W_1 \cap W_2 \cap W_3) \geq 1$. Thus, there exists $0 \neq b \in W_1 \cap W_2 \cap W_3$. Since $W_i = Ann_{\mathcal{B}}(\alpha_i)$, $i = 1, 2, 3$, $\alpha_i b = 0$ for $i = 1, 2, 3$. Thus, $b \in Ann_{\mathcal{B}}(N) = (0)$ which is a contradiction.

Therefore, all four cases are impossible. Hence we conclude that $\mu_{\mathcal{B}}(N) = 2$. □ □

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