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ON FUZZY ALMOST S-CONTINUOUS FUNCTIONS

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Abstract. In this note, the notion of fuzzy almost s-continuity is introduced and some results related to this notion are obtained.

1. Introduction

In [1], B. Ghosh introduced and investigated the notions of fuzzy semi-T_2 spaces and fuzzy semi-connected spaces. In particular, he studied these spaces under fuzzy semi-continuity. T. Noiri, B. Ahmad and M. Khan [2] introduced and studied the notion of almost s-continuous functions. The purpose of this paper is to introduce the notion of fuzzy almost s-continuous functions and to study fuzzy semi-T_2 spaces and fuzzy semi-connected spaces under fuzzy almost s-continuity.

2. Preliminaries

Throughout this paper X and Y will denote fuzzy topological spaces. For definitions and notations which are not explained in this paper, we refer to [1]. For any $\alpha \in (0,1]$ and any $x \in X$, a *fuzzy point* x_α in X is a fuzzy set in X defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{for } y = x \\ 0 & \text{for } y \neq x. \end{cases}$$

A fuzzy point x_α is said to belong to a fuzzy set A in X if $\alpha \leq A(x)$. In this case we shall use the notation $x_\alpha \in A$ [1]. A fuzzy open set [resp. fuzzy semi-open set] U in X is called a *fuzzy open neighborhood* [resp. *fuzzy semi-open neighborhood*] of a fuzzy point x_α in X if $x_\alpha \in U$. Two

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fuzzy sets A and B in X are said to be *q-coincident* (denoted by A_qB) if there exists $x \in X$ such that $A(x) + B(x) > 1$. When two fuzzy sets A and B in X are not q-coincident, we shall write $A_{\not q}B$ [1]. For a fuzzy point x_α in X, a fuzzy set A in X is called a *q-neighborhood* [resp. *semi q-neighborhood*] of x_α if there exists a fuzzy open set [resp. fuzzy semi-open set] U in X such that $x_{\alpha q}U \leq A$ [1]. A fuzzy set is called a *fuzzy semi-regular* if it is both fuzzy semi-open and fuzzy semi-closed. The intersection of all fuzzy semi-closed set containing a fuzzy set A is called the *fuzzy semi-closure* of A and is denoted by $sCl(A)$. The union of all fuzzy semi-open sets contained in a fuzzy set B is called the *fuzzy semi-interior* of B and is denoted by $sInt(B)$.

3. The characterization of fuzzy almost s-continuity

LEMMA 3.1. *If A is fuzzy semi-open in X, then $sCl(A)$ is fuzzy semi-regular in X.*

Proof. By definition, $sCl(A)$ is fuzzy semi-closed, we are left to show that $sCl(A)$ is fuzzy semi-open. Since A is fuzzy semi-open, there exists a fuzzy open set O in X such that $O \leq A \leq Cl(O)$. This implies that $O \leq sCl(O) \leq sCl(A) \leq sCl(Cl(O)) = Cl(O)$ and hence $sCl(A)$ is fuzzy semi-open. □ □

DEFINITION 3.2.. A function $f : X \to Y$ is said to be *fuzzy almost s-continuous* if for each fuzzy point $x_\alpha \in X$ and each fuzzy semi-open set V in Y with $f(x_\alpha) \in V$, there exists a fuzzy open set O in X with $x_\alpha \in O$ such that $f(O) \leq sCl(V)$.

LEMMA 3.3. *A function $f : X \to Y$ is fuzzy almost s-continuous if and only if for any fuzzy semi-regular set A in Y, $f^{-1}(A)$ is both fuzzy open and fuzzy closed in X.*

Proof. Suppose that $f : X \to Y$ is fuzzy almost s-continuous and that A is fuzzy semi-regular in Y. If $f^{-1}(A) = O_X$, then clearly $f^{-1}(A)$ is both fuzzy open and fuzzy closed in X. Let x_α be a fuzzy point in $f^{-1}(A)$. Then $f(x_\alpha) \in A$. By hypothesis, there exists a fuzzy open set O_{x_α} in X with $x_\alpha \in O_{x_\alpha}$ such that $f(O_{x_\alpha}) \leq sCl(A) = A$, and hence we obtain $f^{-1}(A) = \cup\{x_\alpha | x_\alpha \in f^{-1}(A)\} \leq \cup\{O_{x_\alpha} | x_\alpha \in A\} \leq f^{-1}(A)$. This shows that $f^{-1}(A)$ is fuzzy open in X. Now, since $1 - A$ is fuzzy semi-regular in Y, $1 - f^{-1}(A) = f^{-1}(1 - A)$ is fuzzy open in X. Consequently, $f^{-1}(A)$ is fuzzy closed in X.

Conversely, assume that the given condition holds. Let x_α be a fuzzy point in X and let V be a fuzzy semi-open set in Y with $f(x_\alpha) \in V$. By Lemma 3.1, $sCl(V)$ is fuzzy semi-regular. By hypothesis, $f^{-1}(sCl(V))$ is fuzzy open in X with $x_\alpha \in f^{-1}(sCl(V))$. Since $f(f^{-1}(sCl(V))) \leq sCl(V)$, we conclude that f is fuzzy almost s-continuous. □ □

4. Fuzzy semi-T_2 spaces and fuzzy semi-connected spaces

Definition 4.1.. ([1]) A fuzzy topological space X is *fuzzy* T_2 [resp. *fuzzy semi-*T_2] if for every pair of distinct fuzzy points x_α and y_β, the following conditions are satisfied:

(1) If $x \neq y$, then there exist two fuzzy open sets [resp. fuzzy semi-open sets] U and V such that $x_\alpha \in U$, $y_\beta \in V$ and $U \bar{q} V$.

(2) If $x = y$ and $\alpha < \beta$, then x_α has a fuzzy open neighborhood [resp. fuzzy semi-open neighborhood] U and y_β has a q-neighborhood [resp. semi q-neighborhood] V such that $U \bar{q} V$.

Obviously, every fuzzy T_2 space is fuzzy semi-T_2.

Lemma 4.2. *A fuzzy topological space X is fuzzy semi-T_2 if and only if for every pair of distinct fuzzy points x_α and y_β, the following conditions are satisfied:*

(1) If $x \neq y$, then there exist two fuzzy semi-open sets U' and V' such that $x_\alpha \in U'$, $y_\beta \in V'$ and $sCl(U') \bar{q} sCl(V')$.

(2) If $x = y$ and $\alpha < \beta$, then there exist two fuzzy semi-open sets U' and V' such that $x_\alpha \in U', y_\beta q V'$ and $sCl(U') \bar{q} sCl(V')$.

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Assume $x \neq y$. By hypothesis, there exist fuzzy semi-open sets U and V such that $x_\alpha \in U, y_\beta \in V$ and $U \bar{q} V$. Let $U' = sInt(1 - V)$. Clearly, U' is fuzzy semi open in X. Since $U \bar{q} V$, we have $x_\alpha \in U = sInt(U) \leq sInt(1 - V) = U'$. Now, let $V' = 1 - sCl(U')$. Then V' is a fuzzy semi-open set in X. Since $sCl(U') + V = sCl(sInt(1 - V)) + V \leq (1 - V) + V = 1 \leq 1$, we obtain $y_\beta \in V \leq 1 - sCl(U') = V'$. By Lemma 3.1, $sCl(V') = V'$. Since $sCl(U') + sCl(V') = sCl(U') + V' = sCl(U') + (1 - sCl(U')) = 1 \leq 1$, we have $sCl(U') \bar{q} sCl(V')$.

Assume that $x = y$ and $\alpha < \beta$. By hypothesis, x_α has a fuzzy semi-open neighborhood U and y_β has a semi q-neighborhood V such that $U_{\not q}V$. Choose a fuzzy semi-open set W in X such that $y_{\beta q}W \leq V$. Let $V' = sInt(1-U)$. Then V' is fuzzy semi-open in X. Since $U_{\not q}W$ and $y_{\beta q}W$, we have $\beta + V'(y) = \beta + sInt(1-U)(y) \geq \beta + W(y) > 1$. Thus $y_{\beta q}V'$. Now, let $U' = 1 - sCl(V')$. Clearly, U' is a fuzzy semi-open set in X. Since $sCl(V') + U = sCl(sInt(1-U)) + U \leq (1-U) + U = 1 \leq 1$, we have $x_\alpha \in U \leq 1 - sCl(V') = U'$. By Lemma 3.1, $sCl(U') = U'$. Since $sCl(V') + sCl(U') = sCl(V') + U' = sCl(V') + (1 - sCl(V')) = 1 \leq 1$, we obtain $sCl(U')_{\not q}sCl(V')$. □ □

Theorem 4.3. *Let $f : X \to Y$ be injective and fuzzy almost s-continuous. If Y is fuzzy semi-T_2, then X is fuzzy T_2.*

Proof. Let x_α and y_β be two distinct fuzzy points in X.

First, assume that $x \neq y$. Since $f(x) \neq f(y)$, by (1) of Lemma 4.2, there exist two fuzzy semi-open sets U and V in Y such that $f(x)_\alpha \in U, f(y)_\beta \in V$ and $sCl(U)_{\not q}sCl(V)$. This implies that $x_\alpha \in f^{-1}(sCl(U)), y_\beta \in f^{-1}(sCl(V))$ and $f^{-1}(sCl(U))_{\not q}f^{-1}(sCl(V))$. Moreover, by Lemma 3.1 and Lemma 3.3, $f^{-1}(sCl(U))$ and $f^{-1}(sCl(V))$ are fuzzy open in X.

Now, assume that $x = y$ and $\alpha < \beta$. Since $f(x) = f(y)$, by (2) of Lemma 4.2, there exist two fuzzy semi-open sets U and V in Y such that $f(x)_\alpha \in U, f(y)_{\beta q}V$ and $sCl(U)_{\not q}sCl(V)$. Thus, we have $x_\alpha \in f^{-1}(sCl(U)), y_{\beta q}f^{-1}(sCl(V))$ and $f^{-1}(sCl(U))_{\not q}f^{-1}(sCl(V))$. Moreover, by Lemma 3.1 and Lemma 3.3, $f^{-1}(sCl(U))$ and $f^{-1}(sCl(V))$ are fuzzy open in X. □ □

Corollary 4.4. *Let $f : X \to Y$ be injective and fuzzy almost s-continuous. If Y is fuzzy semi-T_2, so is X.*

Definition 4.5. ([1]) Two nonempty fuzzy sets A and B in X are said to be *fuzzy separated* [resp. *fuzzy semi-separated*] if $A_{\not q}Cl(B)$ and $B_{\not q}Cl(A)$ [resp. $A_{\not q}sCl(B)$ and $B_{\not q}sCl(A)$]. A fuzzy topological space which can not be expressed as the union of two fuzzy separated sets [resp. fuzzy semi-separated sets] is said to be *fuzzy connected* [resp. *fuzzy semi-connected*].

LEMMA 4.6. *([1]) Two nonempty fuzzy sets A and B are fuzzy semi-separated if and only if there exist two fuzzy semi-open sets U and V such that $A \leq U, B \leq V, A_{q}V$ and $B_{q}U$.*

LEMMA 4.7. *A fuzzy topological space X is not fuzzy semi-connected if and only if there exist two fuzzy semi-open sets U and V such that $U_{q}V$ and $U \cup V = X$.*

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) Assume that X is not fuzzy semi-connected. Then there exist two fuzzy semi-separated sets A and B in X such that $A \cup B = X$. By Lemma 4.6, it is possible to choose two fuzzy semi-open sets U and V such that $A \leq U, B \leq V, A_{q}V$ and $B_{q}U$. We wish to show that $A = U$ and $B = V$. Note that $B \leq 1-A$ and $A \leq 1-B$. Since $A \cup B = X$, we have that for any $x \in X$, either $A(x) = 1$ or $A(x) = 0$, and $A(x) = 1$ if and only if $B(x) = 0$. Assume that $A(x) = 0$. Then $B(x) = 1$. Since $B_{q}U$, We obtain $U(x) = 0$, and hence $A = U$. Similarly, we obtain $B = V$. □ □

THEOREM 4.8. *Let $f : X \to Y$ be surjective and fuzzy almost s-continuous. If X is fuzzy connected, then Y is fuzzy semi-connected.*

Proof. Suppose to the contrary that Y is not fuzzy semi-connected. By Lemma 4.7, there exist two fuzzy semi-open sets U and V in Y such that $U_{q}V$ and $U \cup V = Y$. This means that $f^{-1}(U)_{q}f^{-1}(V)$ and $f^{-1}(U) \cup f^{-1}(V) = X$. Moreover, U and V are fuzzy semi-regular in Y. By Lemma 3.3, $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy closed in X. This says that X is not fuzzy connected, contrary to the hypothesis. □ □

Since every fuzzy semi -connected set is fuzzy connected, we have

COROLLARY 4.9. *Let $f : X \to Y$ be surjective and fuzzy almost s-continuous. If X is fuzzy connected, so is Y.*

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