Kangweon-Kyungki Math. Jour. **4** (1996), No. 1, pp. 1–5

A STRONG UNIFORM BOUNDEDNESS RESULT ON κ-SPACES

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Abstract. A strong Banach-Mackey property is established for κ-spaces including all complete and some non-complete metric linear spaces and some non-metrizable locally convex spaces. As applications of this result, a strong uniform boundedness result and a new Banach-Steinhaus type theorem are obtained.

A topological vector space X is said to be a κ-space if every null sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that the series $\sum_{k=1}^{\infty} x_{n_k}$ converges in X ([1], [2]). κ-spaces make a large class containing complete metric linear spaces, some non-complete metric linear spaces and some non-metrizable locally convex spaces. Recently, κ-spaces have been shown to enjoy many nice properties ([1],[2],[3],[4]). In this paper, we would like to establish a strong Banach-Mackey property for κ-spaces and derive from this several important results including a strong uniform boundedness principle and a Banach-Steinhaus type theorem.

For the remainder of this note, (X, τ) represents a topological vector space with the vector space X and the vector topology τ on X. Sometimes, we use only X instead of (X, τ). For a topological vector space X let X', X^s and X^b denote the families of continuous, sequentially continuous and bounded linear functionals on X, respectively ([5]). Similarly, for topological vector spaces X and Y let $L(X, Y)$, $SC(X, Y)$ and $B(X, Y)$ denote the families of continuous, sequentially continuous and bounded linear operators from X into Y, respectively. It is easy to see that $X' \subseteq X^s \subseteq X^b$ and $L(X, Y) \subseteq SC(X, Y) \subseteq B(X, Y)$ but, in general, they are different.

Received September 14, 1995.

1991 Mathematics Subject Classification: 46A03, 46B10.

Key words and phrases: κ-spaces, uniform boundedness.

This paper was supported (in part) by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1994

If E is a vector space and F is a vector space of linear functionals on E, then we have a dual pair (E, F). Let $\sigma(E, F)$, $\tau(E, F)$ and $\beta(E, F)$ denote the weak topology, the Mackey topology and the strong topology for E, respectively ([5]). A locally convex space X is said to be a Banach-Mackey space if each bounded set in X is strongly bounded, i.e., $\beta(X, X')$-bounded ([5]). A dual pair (E, F) is called a Banach-Mackey pair if each $\sigma(E, F)$-bounded subset of E is $\beta(E, F)$-bounded. It is easy to see that a dual pair (E, F) is Banach-Mackey if and only if E with every (E, F)-compatible topology is a Banach-Mackey space and, hence, a locally convex space X is Banach-Mackey if and only if (X, X') is a Banach-Mackey pair.

Since $X' \subseteq X^b$ for every locally convex space X, if the dual pair (X, X^b) is Banach-Mackey, then (X, X') must be Banach-Mackey, i.e., X must be a Banach-Mackey space. Moreover, if (X, X^b) is Banach-Mackey, then X has the following more strong property.

PROPOSITION 1. *Let X be a locally convex space. If (X, X^b) is a Banach-Mackey pair, then for every (X, X')-admissible topology τ, (X, τ) is a Banach-Mackey space.*

Proof. It is enough to show that $(X, \beta(X, X'))$ is Banach-Mackey. As was stated above, (X, X') is Banach-Mackey by the assumption and, hence, $\sigma(X, X')$-boundedness $=$ $\beta(X, X')$-boundedness. Let f be a continuous linear functional on $(X, \beta(X, X'))$. Then f sends bounded sets in $(X, \beta(X, X'))$ to bounded sets in $\mathbb{C}$ and, hence, $f \in X^b$ because X with its original topology has the same bounded sets as $(X, \beta(X, X'))$. Now $(X, (X, \beta(X, X'))')$ must be Banach-Mackey because (X, X^b) is Banach-Mackey. □ □

THEOREM 2. *If a locally convex space X is a κ-space, then (X, X^b) is a Banach-Mackey pair.*

Proof. Let $A \subseteq X$ be $\sigma(X, X^b)$-bounded and $B \subseteq X^b$ be $\sigma(X^b, X)$-bounded. Suppose that $\{f(x) : f \in B, x \in A\}$ is not bounded. Then there exist sequences $\{x_j\} \subseteq A$ and $\{f_j\} \subseteq B$ such that $|f_j(x_j)| > j^2$ for all j.

Consider the matrix $[i^{-1} f_i(j^{-1} x_j)]_{i,j}$. For a fixed j, $lim_i i^{-1} f_i(j^{-1} x_j) = 0$ because $\{f_i\} \subseteq B$ and B is $\sigma(X^b, X)$-bounded. Since $\sigma(X, X^b)$-boundedness $=$ $\sigma(X, X')$-boundedness, A is bounded in X by the

Mackey theorem so $j^{-1}x_j \to 0$ in X. Let $\{j_k\}$ be an increasing sequence in $\mathbb{N}$. Then $\{j_k\}$ has a subsequence $\{j_{k_l}\}$ such that the series $\sum_{l=1}^{\infty} j_{k_l}^{-1} x_{j_{k_l}}$ converges in X because X is a κ-space. Now observe that $X^b = X^s$ ([4], Theorem 2) and, hence, each f_i is sequentially continuous on X, $\sum_{l=1}^{\infty} f_i(j_{k_l}^{-1} x_{j_{k_l}}) = f_i(\sum_{l=1}^{\infty} j_{k_l}{}^{-1} x_{j_{k_l}})$ for each i. Thus, $lim_i \sum_{l=1}^{\infty} i^{-1} f_i(j_{k_l}^{-1} x_{j_{k_l}}) = lim_i i^{-1} f_i(\sum_{i=1}^{\infty} j_{k_l}^{-1} x_{j_{k_l}}) = 0$. Now by a matrix theorem, $j^{-1} f_j(j^{-1}x_j) \to 0$ ([3], Theorem 1). This contradicts that $|f_j(x_j)| > j^2$ for all j. Therefore, $\{f(x) : f \in B, x \in A\}$ is bounded. □ □

Corollary 3. *If X is a locally convex κ-space, then X with the strong topology $\beta(X, X')$ is Banach-Mackey.*

Corollary 4. *If X is a locally convex κ-space, then every continuous linear functional on $(X, \beta(X, X'))$ is sequentially continuous on X.*

Proof. As was stated in the proof of Proposition 1, every continuous linear functional on $(X, \beta(X, X'))$ belongs to X^b. But $X^b = X^s$ because X is a κ-space ([4], Theorem 2). □ □

For a locally convex space (X, τ) let τ^b denote the strongest locally convex topology on X which has the same boundedness as τ. (X, τ^b) is bornological and $\tau^b = \tau(X, X^b)$, the Mackey topology in the pair (X, X^b)([5],[6]).

Corollary 5. *If (X, τ) is a locally convex κ-space, then (X, τ^b) is barrelled.*

Proof. (X, τ^b) is bornological and, hence, quasibarrelled ([5], 10-1-10). Since $\tau^b = \tau(X, X^b)$, (X, τ^b) is Banach-Mackey by Theorem 2. Thus, (X, τ^b) must be barrelled ([5], 10-4-12). □ □

A recent result says that $(X^s, \sigma(X^s, X))$ is sequentially complete if X is a κ-space ([4], Theorem 22). We improve this result as follows.

Corollary 6. *If X is a locally convex κ-space, then $(X^s, \sigma(X^s, X))$ is boundedly complete and, hence, sequentially complete, semireflexive and X has the convex compactness property.*

Proof. Observing $X^s = X^b$ and $\tau^b = \tau(X, X^b) = \tau(X, X^s)$ and, therefore, $X^s = (X, \tau^b)'$, $(X^s, \sigma(X^s, X))$ must be boundedly complete by above Corollary 5 and Theorem 9-3-13 of [5]. □ □

There is a nice uniform boundedness result holds for sequentially complete locally convex spaces ([3], Corollary 4 and Proposition 5). We show that the same result holds for locally convex κ-spaces.

THEOREM 7. *Let X be a locally convex κ-space and Y an arbitrary locally convex space. If a subfamily F of $SC(X, Y)$ is pointwise bounded on X, then F is uniformly bounded on bounded subsets of X.*

Proof. Let y' be a continuous linear functional on Y. It is easy to see that $\{y' \circ T : T \in F\} \subseteq X^b$ and $\{y' \circ T : T \in F\}$ is $\sigma(X^b, X)$-bounded. Now let A be a bounded subset of X. Then A is $\sigma(X, X^b)$-bounded. By Theorem 2, the set $\{y'(T(x)) : T \in F, x \in A\}$ is bounded in $\mathbb{C}$, i.e., $\{T(x) : T \in F, x \in A\}$ is $\sigma(Y, Y')$-bounded in Y and, hence, bounded in Y by the Mackey theorem. This just shows that F is uniformly bounded on bounded sets. □ □

Note that above proof is available for the case of $F \subseteq B(X, Y)$, i.e., we can put $B(X, Y)$ instead of $SC(X, Y)$ in Theorem 7. But this is just Theorem 7 itself because $B(X, Y) = SC(X, Y)$ for κ-space X and every locally convex space Y ([4], Corollary 17).

We know the discussions of Banach-Steinhaus type results for non-barrelled spaces are complicated very much ([7],[8]). However, a clear-cut Banach-Steinhaus type result holds for κ-spaces.

THEOREM 8. *Let X be a locally convex κ-space and Y an arbitrary locally convex space. If $\{T_k\}$ is a sequence of sequentially continuous linear operators such that $lim_k T_k x = Tx$ exists in Y for each $x \in X$, then the limit operator T is also a sequentially continuous linear operator.*

Proof. Let $A \subseteq X$ be bounded. Clearly, $\{T_k : k \in \mathbb{N}\}$ is pointwise bounded on X. By Theorem 7, $\{T_k x : x \in A, k \in \mathbb{N}\}$ is bounded and, hence, $\overline{\{T_k x : x \in A, k \in \mathbb{N}\}}$ is bounded in Y. Therefore, $\{Tx : x \in A\}$ is bounded in Y. This shows that $T \in B(X, Y)$. But $B(X, Y) = SC(X, Y)$ ([4], Corollary 17) so T is sequentially continuous. □ □

As an immediate consequence of Theorem 8, we have the following generalization of Theorem 22 of [4].

COROLLARY 9. *If X is a κ-space and Y an arbitrary sequentially complete locally convex space, then $SC(X, Y)$ is sequentially complete with respect to the topology of pointwise convergence on X.*

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