Journal of the Korean Institute
of Industrial Engineers
Vol.22, No.1, March, 1996

Economic Design of a Two-Sided Two-Stage Screening Procedure with a Prescribed Outgoing Quality

Hyuck Moo Kwon*, Do Sun Bai**

Abstract

An economic two-stage screening procedure is presented when both lower and upper specification limits are given on the performance variable. A screening variable which is highly correlated with the performance variable is used first to decide whether an item should be accepted, rejected, or undecided. The performance variable is then used to classify the undecided items. The two variables are assumed to be jointly normally distributed. A cost model is constructed on the basis of six cost components; inspection costs of screening and performance variables and costs caused by type I and type II misclassification errors related with lower and upper specification limits. Optimal cutoff values on the screening variable are determined so that the average outgoing quality exceeds a prespecified level. Solution methods are provided for both known-parameter and unknown-parameter cases.

1. INTRODUCTION

Complete inspections are increasingly attractive in industries due to advances in automatic inspection equipments. In a complete inspection, every item is subject to acceptance inspection and any item failing to meet the predetermined specifications is

* Department of Industrial Engineering, Pusan National University of Technology, Yongdang-dong San 100, Pusan 608-739, Korea

** Department of Industrial Engineering, Korea Advanced Institute of Science and Technology, Gusung-dong 373-1, Yusung-gu, Daejon 305-701, Korea

rejected. Often the major quality characteristic (performance variable) is difficult to measure and a variable which is correlated with the performance variable (screening variable) is used for screening. For example, when the "gain" of an amplifier in an integrated circuit chip is the performance variable of interest, we may choose the "beta" of its monitor transistor as a screening variable.

Most existing studies on the screening procedure can be classified into two groups; i) one is focused on improving the outgoing quality to a prespecified level and ii) the other is focused on reducing the expected cost. See Tang and Tang(1994) for detailed review. In a screening procedure, two kinds of misclassification errors may occur; type I error of rejecting a conforming item and type II error of accepting a nonconforming item. These errors can be reduced by inspecting the performance variable for those items which are difficult to classify correctly by the observed values of the screening variable. Based on this idea, Tang(1988a) proposed an economic two-stage screening procedure using the performance variable as well as the screening variable. Often, both outgoing quality and cost are important. Bai and Kwon (1995) proposed an economic two-stage screening procedure which guarantees a prescribed average outgoing quality (AOQ). The optimal procedures were provided when a single specification limit is given on the performance variable. For many products, however, both lower and upper specification limits are given on the performance variable. Two-sided screening procedures based only on the screening variables were studied by Li and Owen(1979), and Haas et al.(1985), and Tang (1988b).

In this paper, we present an economic two-stage screening procedure with a prescribed AOQ based on both screening and performance variables, extending the results of Bai and Kwon (1995) to the case where both lower and upper specification limits are given on the performance variable. The screening and performance variables are assumed to be jointly normally distributed. A simple cost model is constructed involving six cost components; inspection costs of screening and performance variables and costs caused by type I and type II misclassification errors related with lower and upper specification limits. Optimal cutoff values on the screening variable minimizing the expected cost are obtained subject to the constraint that the AOQ exceeds a prespecified level. The model is described in Section 2 and solution methods for both known-parameter and unknown-parameter cases are presented in Section 3. A numerical example is given in Section 4.

2. THE MODEL

Notations

Y	performance variable
X	screening variable
μ_y, μ_x	means of Y and X
σ_y, σ_x	standard deviations of Y and X
ρ	correlation coefficient of Y and X ($0 << \rho < 1$)
l , u	lower and upper specification limits on Y
ω_{l1}, ω_{l2}, ω_{u1}, ω_{u2}	cutoff values of X ($\omega_{l1} \le \omega_{l2} \le \omega_{u1} \le \omega_{u2}$)
γ	proportion of conforming items before screening
c_y, c_x	unit costs of inspection with the performance and screening variables

c_{rl}, c_{ru}	unit costs caused by the type I misclassification errors in relation to l and u
c_{al}, c_{au}	unit costs caused by type II misclassification errors in relation to l and u
π	proportion of inspection with the performance variable
$\Phi(.)$, $\phi(.)$	distribution function (df) and probability density function (pdf) of standard normal distribution
$G_f(.)$, $g_f(.)$	df and pdf of central t distribution with f degrees of freedom

The Design Problem

Consider the situation where there are both lower and upper specification limits on the performance variable Y; items with $l \le Y \le u$ are conforming and those with $Y < l$ or $Y > u$ are nonconforming. Assume that the screening variable X and Y have a bivariate normal distribution with means μ_x and μ_y, standard deviations σ_x and σ_y, and positive correlation coefficient ρ. Then the two-stage screening procedure is :

First Stage : Take a measurement x of X for each incoming item. The item is (a) rejected if $x < \omega_{l1}$ *or* $x > \omega_{u2}$, *(b) undecided if* $\omega_{l1} \le x \le \omega_{l2}$ *or* $\omega_{u1} \le x \le \omega_{u2}$, *and (c) accepted if* $\omega_{l2} < x < \omega_{u1}$.

Second Stage : Take a measurement y of Y for each undecided item in the first stage. It is (a) rejected if $y < l$ *or* $y > u$, *and (b) accepted if* $l \le y \le u$.

Here, ω_{l1}, ω_{l2}, ω_{u1}, and ω_{u2} are the cutoff values to be determined. If X and Y are negatively correlated, the same procedure will be still valid by replacing X with $-X$.

The average outgoing quality(AOQ) and the expected total cost (ETC) per item for this procedure are obtained by

$$AOQ = \frac{\gamma - \varepsilon_{l1} - \varepsilon_{u1}}{\gamma - \varepsilon_{l1} - \varepsilon_{u1} + \varepsilon_{l2} + \varepsilon_{u2}}, \tag{1}$$

$$ETC = c_{rl}\varepsilon_{l1} + c_{ru}\varepsilon_{u1} + c_{al}\varepsilon_{l2} + c_{au}\varepsilon_{u2} + c_y\pi + c_x, \tag{2}$$

where $\gamma = P(l \le Y \le u)$, $\varepsilon_{l1} = P(X < \omega_{l1}, l \le Y \le u)$, $\varepsilon_{u1} = P(X > \omega_{u2}, l \le Y \le u)$, $\varepsilon_{l2} = P(\omega_{l2} < X < \omega_{u1}, Y < l)$, $\varepsilon_{u2} = P(\omega_{l2} < X < \omega_{u1}, Y > u)$, $\pi = P(\omega_{l1} \le X \le \omega_{l2}) + P(\omega_{u1} \le X \le \omega_{u2})$ and the cost components are defined in the Notations.

The design problem is to find (ω_{l1}, ω_{l2}, ω_{u1}, ω_{u2}) which minimizes ETC subject to $AOQ \ge \delta$.

3. OPTIMAL SOLUTIONS

Case of Known Parameters

When all parameters are known, l, u, ω_{li}, and ω_{ui}, $i = 1, 2$, can be standardized as $\tau_l = (l - \mu_y)/\sigma_y$, $\tau_u = (u - \mu_y)/\sigma_y$, $k_{li} = (\omega_{li} - \mu_x)/\sigma_x$, and $k_{ui} = (\omega_{ui} - \mu_x)/\sigma_x$, respectively. After determining the optimal values k_{li}^* and k_{ui}^*, the optimal cutoff values can be obtained simply by $\omega_{li}^* = \mu_x + k_{li}^*\sigma_x$ and $\omega_{ui}^* = \mu_x + k_{ui}^*\sigma_x$, $i = 1, 2$.

Note that, in a screening procedure with ρ reasonably close to 1, the probabilities $P(X \ge \omega_{u1}, Y < l)$ and $P(X \le \omega_{l2}, Y > u)$ will be nearly zero or negligible and ε_{l1}, ε_{u1}, ε_{l2} and ε_{u2} can be approximated by $P(X < \omega_{l1}, Y \ge l)$, $P(X > \omega_{u2}, Y \le u)$, $P(X > \omega_{l2}, Y < l)$ and $P(X < \omega_{u1}, Y > u)$, respectively. These probabilities can be rewritten as functions

$\alpha_l(k_{l1})$, $\alpha_u(k_{u2})$, $\beta_l(k_{l2})$, and $\beta_u(k_{u1})$ of k_{l1}, k_{u2}, k_{l2} and k_{u1}, respectively. That is,

$$\alpha_l(k_{l1}) = \int_{-\infty}^{k_{l1}} \left\{ 1 - \Phi\left(\frac{\tau_l - \rho z}{\sqrt{1-\rho^2}} \right) \right\} \phi(z) dz, \tag{3a}$$

$$\alpha_u(k_{u2}) = \int_{k_{u2}}^{\infty} \Phi\left(\frac{\tau_u - \rho z}{\sqrt{1-\rho^2}} \right) \phi(z) dz, \tag{3b}$$

$$\beta_l(k_{l2}) = \int_{k_{l2}}^{\infty} \Phi\left(\frac{\tau_l - \rho z}{\sqrt{1-\rho^2}} \right) \phi(z) dz, \tag{3c}$$

$$\beta_u(k_{u1}) = \int_{-\infty}^{k_{u1}} \left\{ 1 - \Phi\left(\frac{\tau_u - \rho z}{\sqrt{1-\rho^2}} \right) \right\} \phi(z) dz. \tag{3d}$$

Now, the design problem is to minimize

$$ETC = c_{rl}\alpha_l(k_{l1}) + c_{ru}\alpha_u(k_{u2}) + c_{al}\beta_l(k_{l2}) + c_{au}\beta_u(k_{u1}) + c_y\pi_k + c_s \tag{4}$$

subject to the constraints $AOQ \geq \delta$ and $k_{l1} \leq k_{l2} \leq k_{u1} \leq k_{u2}$, where

$$\pi_k = \Phi(k_{l2}) - \Phi(k_{l1}) + \Phi(k_{u2}) - \Phi(k_{u1}). \tag{5}$$

This can be solved by a Lagrangean constrained minimization method. Since the requirement $AOQ \geq \delta$ is equivalent to

$$(1/\delta - 1)[\gamma - \alpha_l(k_{l1}) - \alpha_u(k_{u2})] - \beta_l(k_{l2}) - \beta_u(k_{u1}) \geq 0, \tag{6}$$

we obtain the Lagrangean function as

$$\begin{aligned} L = ETC &- \lambda_1 \{ (1/\delta - 1)[\gamma - \alpha_l(k_{l1}) - \alpha_u(k_{u2})] - \beta_l(k_{l2}) - \beta_u(k_{u1}) - s_1^2 \} \\ &- \lambda_2 (k_{l2} - k_{l1} - s_2^2) \\ &- \lambda_3 (k_{u1} - k_{l2} - s_3^2) \\ &- \lambda_4 (k_{u2} - k_{u1} - s_4^2), \end{aligned} \tag{7}$$

where λ_i and s_i, $i = 1,2,3,4$, are Lagrange multipliers and slack variables, respectively. Equating the first derivatives of L with respect to λ_i, s_i, $i = 1,2,3,4$, k_{lj} and k_{uj}, $j = 1, 2$, to zero, we obtain

$$-(1/\delta - 1)[\gamma - \alpha_l(k_{l1}) - \alpha_u(k_{u2})] + \beta_l(k_{l2}) + \beta_u(k_{u1}) + s_1^2 = 0, \tag{8a}$$

$$-k_{l2} + k_{l1} + s_2^2 = 0, \tag{8b}$$

$$-k_{u1} + k_{l2} + s_3^2 = 0, \tag{8c}$$

$$-k_{u2} + k_{u1} + s_4^2 = 0, \tag{8d}$$

$$2\lambda_i s_i = 0, \qquad i = 1,2,3,4, \tag{8e}$$

$$[c_{rl} + \lambda_1(1/\delta - 1)]\phi(k_{l1})\left\{1 - \frac{c_y}{c_{rl} + \lambda_1(1/\delta - 1)} - \Phi\left(\frac{\tau_l - \rho k_{l1}}{\sqrt{1-\rho^2}}\right)\right\} + \lambda_2 = 0, \tag{8f}$$

$$(c_{al} + \lambda_1)\phi(k_{l2})\left\{\frac{c_y}{c_{al} + \lambda_1} - \Phi\left(\frac{\tau_l - \rho k_{l2}}{\sqrt{1-\rho^2}}\right)\right\} - \lambda_2 + \lambda_3 = 0, \tag{8g}$$

$$(c_{au} + \lambda_1)\phi(k_{u1})\left\{1 - \frac{c_y}{c_{au} + \lambda_1} - \Phi\left(\frac{\tau_u - \rho k_{u1}}{\sqrt{1-\rho^2}}\right)\right\} - \lambda_3 + \lambda_4 = 0, \tag{8h}$$

$$[c_{ru} + \lambda_1(1/\delta - 1)]\phi(k_{u2})\left\{\frac{c_y}{c_{ru} + \lambda_1(1/\delta - 1)} - \Phi\left(\frac{\tau_u - \rho k_{u2}}{\sqrt{1-\rho^2}}\right)\right\} - \lambda_4 = 0. \tag{8i}$$

Depending on the values of s_i^*, $i = 2,3,4$, the solutions of these equations may yield one of the following procedures; (i) a typical two-stage screening procedure with $k_{l1}^* < k_{l2}^* < k_{u1}^* < k_{u2}^*$, (ii) a single-stage screening procedure based only on the screening variable with $k_{l1}^* = k_{l2}^* < k_{u1}^* = k_{u2}^*$, (iii) a two-stage screening procedure with $k_{l1}^* = k_{l2}^* < k_{u1}^* < k_{u2}^*$, where an item is rejected if $x < \omega_l^*$ $(= \omega_{l1}^* = \omega_{l2}^*)$ or $x > \omega_{u2}^*$, accepted if $\omega_l^* \le x \le \omega_{u1}^*$, and inspected with the performance variable if $\omega_{u1}^* < x \le \omega_{u2}^*$, (iv) a two-stage screening procedure with $k_{l1}^* < k_{l2}^* < k_{u1}^* = k_{u2}^*$, where an item is rejected if $x < \omega_{l1}^*$ or $x > \omega_u^*$ $(= \omega_{u1}^* = \omega_{u2}^*)$, accepted if $\omega_{l2}^* \le x \le \omega_u^*$, and inspected with the performance variable if $\omega_{l1}^* \le x < \omega_{l2}^*$, (v) a two-stage screening procedure with $k_{l1}^* = k_{l2}^* = k_{u1}^* < k_{u2}^*$, $k_{l1}^* < k_{l2}^* = k_{u1}^* = k_{u2}^*$, or $k_{l1}^* < k_{l2}^* = k_{u1}^* < k_{u2}^*$ where every item which is not rejected in the first stage is inspected with the performance variable in the second stage, and (vi) a trivial procedure with $k_{l1}^* = k_{l2}^* = k_{u1}^* = k_{u2}^*$ where all items are rejected without inspection.

When $s_2^* \neq 0$, $s_3^* \neq 0$ and $s_4^* \neq 0$, procedure (i) is optimal since $k_{l1}^* < k_{l2}^* < k_{u1}^* < k_{u2}^*$ from (8b), (8c) and (8d). By (8e), $\lambda_2^* = \lambda_3^* = \lambda_4^* = 0$ and the optimal solutions are obtained from (8f), (8g), (8h), and (8i) as

$$k_{l1}^* = \frac{1}{\rho}\left[\tau_l + \Phi^{-1}\left(\frac{c_y}{c_{rl} + \lambda_1^*(1/\delta - 1)}\right)\sqrt{1-\rho^2}\right], \tag{9a}$$

$$k_{l2}^* = \frac{1}{\rho}\left[\tau_l - \Phi^{-1}\left(\frac{c_y}{c_{al} + \lambda_1^*}\right)\sqrt{1-\rho^2}\right], \tag{9b}$$

$$k_{u1}^* = \frac{1}{\rho}\left[\tau_u + \Phi^{-1}\left(\frac{c_y}{c_{au} + \lambda_1^*}\right)\sqrt{1-\rho^2}\right], \tag{9c}$$

$$k_{u2}^* = \frac{1}{\rho}\left[\tau_u - \Phi^{-1}\left(\frac{c_y}{c_{ru} + \lambda_1^*(1/\delta - 1)}\right)\sqrt{1-\rho^2}\right], \tag{9d}$$

where λ_1^* is determined so that $(1/\delta - 1)\left[\gamma - \alpha_l(k_{l1}^*) - \alpha_u(k_{u2}^*)\right] - \beta_l(k_{l2}^*) - \beta_u(k_{u1}^*) \geq 0$ and $k_{l1}^* \leq k_{l2}^* \leq k_{u1}^* \leq k_{u2}^*$. See the Appendix for a proof of the optimality. Note that, for any given set of parameters and cost components, the value of λ_1^* can be determined numerically by increasing λ_1 from zero until the requirements $AOQ \geq \delta$ and $k_{l1} \leq k_{l2} \leq k_{u1} \leq k_{u2}$ are satisfied.

When $s_3^* \neq 0$ and $s_2^* = s_4^* = 0$, $k_{l1}^* = k_{l2}^*$ and $k_{u1}^* = k_{u2}^*$ from (8b) and (8d), respectively, and procedure (ii) is optimal. In this case, $\lambda_2^* \neq 0$, $\lambda_3^* = 0$, and $\lambda_4^* \neq 0$ and the optimal solutions can be obtained by eliminating λ_2^* in (8f) and (8g) and λ_4^* in (8h) and (8i) as

$$k_{l1}^* = k_{l2}^* = \frac{1}{\rho}\left[\tau_l - \Phi^{-1}\left(\frac{c_{rl} + \lambda_1^*(1/\delta - 1)}{c_{rl} + c_{al} + \lambda_1^*/\delta}\right)\sqrt{1-\rho^2}\right], \tag{10a}$$

$$k_{u1}^* = k_{u2}^* = \frac{1}{\rho}\left[\tau_u - \Phi^{-1}\left(\frac{c_{au} + \lambda_1^*}{c_{ru} + c_{au} + \lambda_1^*/\delta}\right)\sqrt{1-\rho^2}\right]. \tag{10b}$$

Procedure (iii) is optimal when $s_2^* = 0$, $s_3^* \neq 0$, and $s_4^* \neq 0$, procedure (iv) is optimal when $s_2^* \neq 0$, $s_3^* \neq 0$, and $s_4^* = 0$, and procedure (v) is optimal when $s_2^* = s_3^* = 0$ and

$s_1^* \neq 0$, or $s_2^* \neq 0$ and $s_3^* = s_4^* = 0$, or $s_2^* \neq 0$, $s_3^* = 0$, and $s_4^* \neq 0$. And the trivial procedure (vi) is optimal when $s_2^* = s_3^* = s_4^* = 0$. Formulas (9a) through (9d) or (10a) and (10b) can also be used in obtaining the optimal solutions for these procedures.

Case of Unknown Parameters

When all the parameters of the bivariate normal distribution are unknown, the usual estimators $\overline{X}$, $\overline{Y}$, S_x, S_y, and r are used in place of μ_x, μ_y, σ_x, σ_y, and ρ, respectively. If a preliminary sample $\{(X_1,Y_1), (X_2,Y_2), ..., (X_n,Y_n)\}$ from the unscreened population and vague prior informations on the unknown parameters are available, the predictive distribution of $T_1 = (X-\overline{X})/(\eta S_x)$ and $T_2 = (Y-\overline{Y})/(\eta S_y)$ is bivariate t with joint density function

$$f(t_1,t_2) = \frac{1}{2\pi\sqrt{1-r^2}}\left[1+\frac{t_1^2-2rt_1t_2+t_2^2}{(n-2)(1-r^2)}\right]^{-\frac{n}{2}}, \tag{11}$$

where $\eta = \left[(n-1)(n+1)/n(n-2)\right]^{1/2}$. Each marginal predictive distribution of T_1 or T_2 is central t with $n-2$ degrees of freedom. See Boys and Dunsmore(1986), and Kim and Bai(1992).

Let $\tau_{lp} = (l-\bar{y})/(\eta s_y)$ and $\tau_{up} = (u-\bar{y})/(\eta s_y)$. Define $H_l(t_1) = P(T_2 \geq \tau_{lp} | T_1 = t_1)$ and $H_u(t_1) = P(T_2 \geq \tau_{up} | T_1 = t_1)$. Then $H_l(t_1)$ is

$$\begin{aligned} H_l(t_1) &= \frac{1}{2} I_{a(t_1)}\left((n-1)/2, 1/2\right), && \text{if } t_1 \leq \tau_{lp}/r, \\ &= 1-\frac{1}{2} I_{a(t_1)}\left((n-1)/2, 1/2\right), && \text{if } t_1 > \tau_{lp}/r, \end{aligned} \tag{12}$$

where $a(t_1) = (1-r^2)(n-2+t_1^2)/\left[(1-r^2)(n-2+t_1^2)+(\tau_{lp}-rt_1)^2\right]$, $I_x(\kappa,\nu) = \left[B(\kappa,\nu)\right]^{-1}\int_0^x w^{\kappa-1}(1-w)^{\nu-1}dw$, and $B(\kappa,\nu) = \Gamma(\kappa)\Gamma(\nu)/\Gamma(\kappa+\nu)$. See Bai and Kwon

(1995) for detailed derivations. $H_u(t_1)$ is given by (12) with τ_{up} in place of τ_{lp}.

Let $\xi_{li}=(\omega_{li}-\bar{x})/(\eta s_x)$ and $\xi_{ui}=(\omega_{ui}-\bar{x})/(\eta s_x)$, $i=1,2$. The probabilities $\alpha_l(k_{l1})$, $\alpha_u(k_{u2})$, $\beta_l(k_{l2})$, and $\beta_u(k_{u1})$ are now replaced with the predictive probabilities

$$\alpha_{lp}(\xi_{l1})=\int_{-\infty}^{\xi_{l1}}H_l(t_1)g_{n-2}(t_1)dt_1, \tag{13a}$$

$$\alpha_{up}(\xi_{u2})=1-G_{n-2}(\xi_{u2})-\int_{\xi_{u2}}^{\infty}H_u(t_1)g_{n-2}(t_1)dt_1, \tag{13b}$$

$$\beta_{lp}(\xi_{l2})=1-G_{n-2}(\xi_{l2})-\int_{\xi_{l2}}^{\infty}H_l(t_1)g_{n-2}(t_1)dt_1, \tag{13c}$$

$$\beta_{up}(\xi_{u1})=\int_{-\infty}^{\xi_{u1}}H_u(t_1)g_{n-2}(t_1)dt_1, \tag{13d}$$

respectively. Thus, the equation (4) and the inequality (6) become

$$ETC_p=c_{rl}\alpha_{lp}(\xi_{l1})+c_{ru}\alpha_{up}(\xi_{u2})+c_{al}\beta_{lp}(\xi_{l2})+c_{au}\beta_{up}(\xi_{u1})+c_y\pi_p+c_s, \tag{14}$$

$$(1/\delta-1)\left[\gamma_p-\alpha_{lp}(\xi_{l1})-\alpha_{up}(\xi_{u2})\right]-\beta_{lp}(\xi_{l2})-\beta_{up}(\xi_{u1})\ge 0, \tag{15}$$

respectively, where $\gamma_p=G_{n-2}(-\tau_{lp})+G_{n-2}(\tau_{up})-1$ and $\pi_p=G_{n-2}(\xi_{l2})-G_{n-2}(\xi_{l1})$ $+G_{n-2}(\xi_{u2})-G_{n-2}(\xi_{u1})$. The design problem is now to minimize ETC_p subject to the constraints (15) and $\xi_{l1}\le\xi_{l2}\le\xi_{u1}\le\xi_{u2}$.

By the Lagrangean constrained minimization method used in the case of known parameters, the optimal solutions are obtained as

$$\xi_{l1}^*=H_l^{-1}\left(c_y/\left[c_{rl}+\lambda_1^*(1/\delta-1)\right]\right), \tag{16a}$$

$$\xi_{l2}^*=H_l^{-1}\left(1-c_y/(c_{al}+\lambda_1^*)\right), \tag{16b}$$

$$\xi_{u1}^*=H_u^{-1}\left(c_y/(c_{au}+\lambda_1^*)\right) \tag{16c}$$

$$\xi_{u2}^*=H_u^{-1}\left(1-c_y/\left[c_{ru}+\lambda_1^*(1/\delta-1)\right]\right), \tag{16d}$$

where λ_1^* is determined by increasing λ_1 from zero until the constraints (15) and $\xi_{l1}\le\xi_{l2}\le\xi_{u1}\le\xi_{u2}$ are satisfied. In particular, when $\xi_{l1}=\xi_{l2}<\xi_{u1}=\xi_{u2}$, the optimal solutions are obtained by

$$\xi_{l1}^*=\xi_{l2}^*=H_l^{-1}\left(\frac{c_{al}+\lambda_1^*}{c_{rl}+c_{al}+\lambda_1^*/\delta}\right), \tag{17a}$$

$$\xi_{u1}^{*} = \xi_{u2}^{*} = H_u^{-1}\left(\frac{c_{ru} + \lambda_1^{*}(1/\delta - 1)}{c_{ru} + c_{au} + \lambda_1^{*}/\delta}\right). \tag{18b}$$

4. A Numerical Example

In this section, a numerical example which originally appeared in Li and Owen (1979) is given to illustrate the proposed two-sided two-stage screening procedure. Based on this example, some numerical studies are performed to compare the single- and two-stage screening procedures. For unknown-parameter case, the effects of estimation errors on the true ETC and AOQ are also investigated. IMSL (1987) subroutines are used to evaluate statistical distribution functions and integrations.

An Example. Suppose that the voltage Y at an internal point of an electronic device is its major quality characteristic (performance variable). Since this voltage is difficult to measure directly, the voltage X at an external point (screening variable) is first measured and then Y is measured for only those devices which are difficult to classify correctly by the observed values of X. The lower and upper specification limits are $l = 12$ volts and $u = 16$ volts. The distribution parameters are $\mu_x = 10$ volts, $\mu_y = 13.8$ volts, $\sigma_x = 2$ volts, $\sigma_y = 2.13$ volts and $\rho = 0.90$ and the outgoing quality after screening is desired to exceed $\delta = 0.95$. Let the cost components be $c_x = \$0.05$, $c_y = \$1.00$, $c_{rl} = c_{ru} = \$2.00$, $c_{al} = \$3.00$ and $c_{au} = \$4.00$.

Solution. The proportion of conforming items in the unscreened population is $\gamma = 0.80 + 0.85 - 1 = 0.65$. Using formula (9a) through (9d), we obtain $\lambda_1^{*} = 2.6031$,

$k_{l1}^* = -0.9779$, $k_{l2}^* = -0.4928$, $k_{u1}^* = 0.6486$ and $k_{u2}^* = 1.1866$. The optimal cutoff values are thus $\omega_{l1}^* = 8.04$, $\omega_{l2}^* = 9.01$, $\omega_{u1}^* = 11.30$, and $\omega_{u2}^* = 12.37$. The proportion π of direct inspection with the performance variable is about 28.8% and the expected cost is $ETC = \$0.5564$. The optimal cutoff values of the single-stage procedure based only on the screening variable, which can be obtained using formula (10a) and (10b), are $\omega_1^* = 9.43$ and $\omega_2^* = 10.97$ and $ETC = \$0.8359$. Compared to the single-stage procedure, the cost reduction of the two-stage procedure is about 33.4%.

If all the parameters are unknown, the optimal cutoff values are obtained using formula (16a) through (16d). Suppose that the estimates for the unknown parameters are $\bar{x} = 10$ volts, $\bar{y} = 13.8$ volts, $s_x = 2$ volts, $s_y = 2.13$ volts, and $r = 0.90$ from a preliminary sample of size 12. We then obtain $\lambda_1^* = 4.1504$, $\xi_{l1}^* = -0.9210$, $\xi_{l2}^* = -0.3322$, $\xi_{u1}^* = 0.4763$, and $\xi_{u2}^* = 1.1133$. The optimal cutoff values are thus $\omega_{l1}^* = 7.99$, $\omega_{l2}^* = 9.27$, $\omega_{u1}^* = 11.04$, and $\omega_{u2}^* = 12.43$. The predictive proportion of inspection with the performance variable is $\pi_p = 0.3595$ and the predictive ETC is $ETC_p = \$0.6264$.

Comparison of Single- and Two-stage Procedures. In a two-stage screening procedure, the proportion π will be largely affected by ρ and c_y. Figure 1 shows graphs of π_k versus c_y for $\rho = 0.80(0.05)0.95$. As expected, π_k is small if ρ and c_y are large.

When both lower and upper specification limits are given on the performance variable, for some combinations of values of ρ, δ, and γ, the requirement $AOQ \geq \delta$ cannot be achieved by any single-stage procedure based only on the screening variable. For example, we find no procedures for $(\rho, \delta, \gamma_1) = (0.80, 0.95, 0.80)$ and $(\rho, \delta, \gamma_2) = (0.80, 0.95, 0.85)$ in the tables provided by Li and Owen (1979). In fact, AOQ of any single-stage screening procedure cannot exceed

$$AOQ_{\max} = 2\Phi\left((\tau_u - \tau_l)/2\sqrt{1-\rho^2}\right) - 1 \cdot \tag{19}$$

See the Appendix for a proof. In a two-stage screening procedure, however, the misclassification errors can be reduced to any desired level by increasing the proportion of inspection with the performance variable and the prescribed AOQ can always be attained.

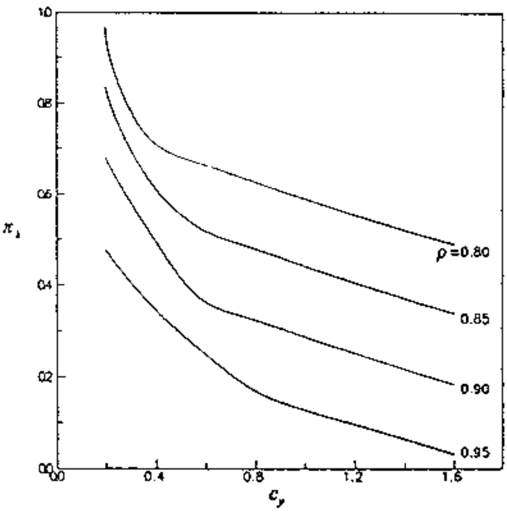


Figure 1. Graphs of π_s for $\rho = 0.80(0.05)0.95$

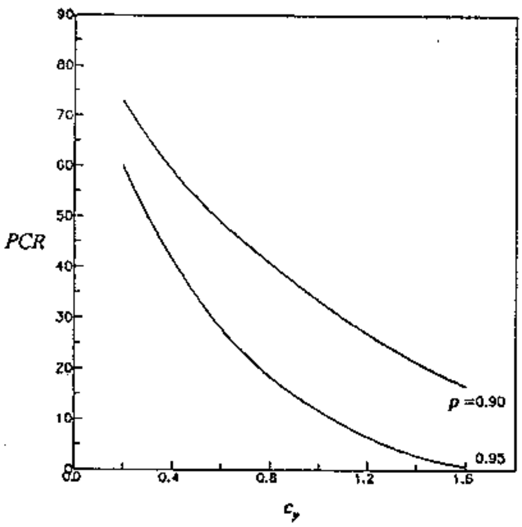


Figure 2. Graphs of PCR for $\rho = 0.90$ and 0.95

Let ETC_s and ETC_t be the expected costs per item of single-stage and two-stage screening procedures, respectively. The percentage cost reduction (PCR) of the two-stage procedure, compared with the single-stage procedure, is then

$$PCR = 100 \times (ETC_s - ETC_t)/ETC_s \cdot \tag{20}$$

In Figure 2, graphs of PCR versus c_y are depicted for $\rho = 0.90$ and 0.95. (There are no single-stage procedures achieving $AOQ \geq 0.95$ when $\rho = 0.80$ or 0.85 in the example.) It seems that a single-stage screening procedure is preferred when the correlation coefficient is close to 1 and the unit cost of inspection with the performance variable is large. However, the percentage cost reduction is over 10% even when $\rho = 0.95$ and $c_v = \$1.0$ which is fairly large compared with the unit price of the item that will be usually near $c_{rl} = c_{ru} = \$2.0$. The smaller the unit cost of inspection of the performance variable is, the larger is the percentage cost reduction.

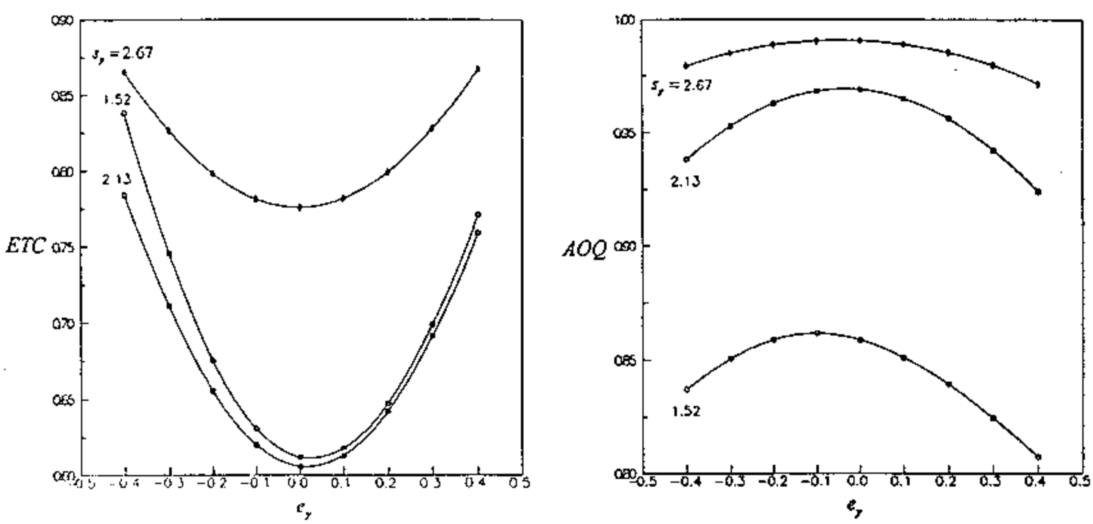
Effects of Estimation Errors. In the above example of unknown-parameter case, we obtained the optimal two-stage screening procedure with $ETC_p = \$0.6264$ and $AOQ_p = 0.95$. However, ETC_p and AOQ_p are obtained based on the estimates of the unknown parameters and may be different from the true ETC and AOQ. For example, in the unlikely case where the unknown parameters are estimated without any errors, that is, when $\mu_x = 10$, $\mu_y = 13.8$, $\sigma_x = 2$, $\sigma_y = 2.13$ and $\rho = 0.90$, the two-stage procedure with cutoff values $\omega_{l1} = 7.99$, $\omega_{l2} = 9.27$, $\omega_{u1} = 11.04$, and $\omega_{u2} = 12.43$ would yield true proportion of inspection with the performance variable $\pi = \Phi((9.27-10)/2)$ $-\Phi((7.99-10)/2) + \Phi((12.43-10)/2) - \Phi((11.04-10)/2)$ = 0.3932 and the true probabilities corresponding to misclassification errors $\alpha_l((7.99-10)/2) = 0.0285$, $\alpha_u((12.43-10)/2)$ $= 0.0215$, $\beta_l((9.27-10)/2) = 0.0108$, and $\beta_u((11.04-10)/2)$ $= 0.0083$, and thus ETC = \$0.6088 < $\$0.6264 = ETC_p$ and AOQ = 0.9691 > $0.95 = AOQ_p$. This implies that the prescribed AOQ is usually guaranteed at a cost lower than ETC_p if the estimation errors are small. When the estimation errors are large, however, the prescribed AOQ may not be attained.

Since there must be sufficient information about ρ before X is selected as a screening variable and μ_x and σ_x can be estimated with sufficient accuracy without bearing expensive cost, we assume that μ_x, σ_x and ρ are correctly estimated and study here only the effects of using incorrect estimates for μ_y and σ_y. We first find the optimal procedures corresponding to various values of $\bar{y}$ and s_y and then calculate the true ETC and AOQ for each procedure assuming that the true values of the unknown parameters are given as in the example. In Figure 3, the true ETC and AOQ obtained by such methods are graphically illustrated as functions of the standardized estimation error $e_y = (\bar{y} - 13.8)/2.13$ of μ_y for $s_y = 1.52$, 2.13, and 2.67. The values 1.52 and 2.67 are selected to investigate the case where σ_y is badly underestimated or overestimated. Note that $P(S_y \leq 1.52)$ or $P(S_y \geq 2.67)$ is at most 0.1. Those graphs show that :

i) When σ_y is overestimated with $s_y = 2.67$, AOQ is higher than the prespecified level $\delta = 0.95$ but ETC is considerably larger than ETC with $s_y = 2.13$.

ii) When σ_y is underestimated with $s_y = 1.52$, the prescribed AOQ may not be attained while ETC is close to ETC with $s_y = 2.13$.

iii) ETC is very sensitive to e_y when σ_y is not overestimated.



(a) Graphs of ETC

(b) Graphs of AOQ

Figure 3. Graphs of true ETC and AOQ for s_y = 1.52, 2.13 and 2.67

iv) AOQ seems to be less sensitive to e_y and more affected by s_y. And the prescribed AOQ is usually guaranteed if σ_y is not underestimated.

5. CONCLUDING REMARKS

We have presented an economic two-sided, two-stage screening procedure which guarantees the AOQ to exceed a prespecified level. A simple cost model is assumed involving costs of inspections and costs due to type I and type II misclassification errors. Assuming bivariate normal probability structure for the two variables, solution procedures are provided for both known-parameter and unknown-parameter cases. When parameters are unknown, a preliminary sample is assumed to be available from the unscreened population. No closed form solutions are obtainable but the optimal cutoff values can be found by numerical search method. Existing softwares such as IMSL (1987) subroutines can be used to obtain the optimal solutions.

The prescribed AOQ for the two-sided case may not be attained in the single-stage screening procedure. In a two-stage screening procedure, however, the misclassification errors can be reduced to any desired levels by increasing the proportion of inspection with the performance variable and the prescribed AOQ can be always attained.

As special cases of the proposed model, optimal cutoff values of the single-stage screening procedure based only on the screening variable and of the two-stage screening procedure with no requirement on the outgoing quality can be obtained. Numerical studies show that the two-stage screening procedure guarantees the prescribed outgoing quality at a considerably lower cost than the single-stage screening procedure even with fairly large c_y and ρ. When parameters are unknown, the prescribed outgoing quality is usually guaranteed provided that σ_y is not severely underestimated.

APPENDICES

I. Proof of Optimality of (9)

To see that the cutoff coefficients given by (9a) through (9d) are optimal, we obtain the Hessian matrix evaluated at $(\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*, s_1^*, s_2^*, s_3^*, s_4^*, k_{l1}^*, k_{l2}^*, k_{u1}^*, k_{u2}^*)$. Let

$$A_1 = (1/\delta - 1)\phi(k_{l1}^*)\left[1 - \Phi\left((\tau_l - \rho k_{l1}^*)/\sqrt{1-\rho^2}\right)\right],$$
$$A_2 = \phi(k_{l2}^*)\Phi\left((\tau_l - \rho k_{l2}^*)/\sqrt{1-\rho^2}\right),$$
$$A_3 = \phi(k_{u1}^*)\left[1 - \Phi\left((\tau_u - \rho k_{u1}^*)/\sqrt{1-\rho^2}\right)\right],$$
$$A_4 = (1/\delta - 1)\phi(k_{u2}^*)\Phi\left((\tau_u - \rho k_{u2}^*)/\sqrt{1-\rho^2}\right),$$
$$B_1 = \left[c_{rl} + \lambda_1^*(1/\delta - 1)\right]\phi(k_{l1}^*)\phi\left((\tau_l - \rho k_{l1}^*)/\sqrt{1-\rho^2}\right)\rho/\sqrt{1-\rho^2},$$
$$B_2 = (c_{al} + \lambda_1^*)\phi(k_{l2}^*)\phi\left((\tau_l - \rho k_{l2}^*)/\sqrt{1-\rho^2}\right)\rho/\sqrt{1-\rho^2},$$
$$B_3 = (c_{au} + \lambda_1^*)\phi(k_{u1}^*)\phi\left((\tau_u - \rho k_{u1}^*)/\sqrt{1-\rho^2}\right)\rho/\sqrt{1-\rho^2},$$
$$B_4 = \left[c_{ru} + \lambda_1^*(1/\delta - 1)\right]\phi(k_{u2}^*)\phi\left((\tau_u - \rho k_{u2}^*)/\sqrt{1-\rho^2}\right)\rho/\sqrt{1-\rho^2}.$$

Then the Hessian matrix H is given by

$$\mathrm{H} = \begin{bmatrix} \mathrm{O} & \Sigma & \mathrm{A} \\ \Sigma & \Lambda & \mathrm{O} \\ \mathrm{A}' & \mathrm{O} & \mathrm{B} \end{bmatrix},$$

where

$$\mathrm{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 2s_1^* & 0 & 0 & 0 \\ 0 & 2s_2^* & 0 & 0 \\ 0 & 0 & 2s_3^* & 0 \\ 0 & 0 & 0 & 2s_4^* \end{bmatrix}, \mathrm{A} = \begin{bmatrix} A_1 & -A_2 & A_3 & -A_4 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 2\lambda_1^* & 0 & 0 & 0 \\ 0 & 2\lambda_2^* & 0 & 0 \\ 0 & 0 & 2\lambda_3^* & 0 \\ 0 & 0 & 0 & 2\lambda_4^* \end{bmatrix}, \mathrm{B} = \begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix}.$$

Denote the j^{th} principal minor determinant of H by $\det(H^j)$. From the fact that $A_i \geq 0$, $B_i \geq 0$, and $\lambda_i^* \geq 0$, $i = 1,2,3,4$,

$$\det(H^j) = 0 \text{ for } j = 1,2,\ldots,7,$$

$$\det(H^8) = 256\prod_{i=1}^{4} s_i^{*2} \geq 0,$$

$$\det(H^9) = 128(2B_1 s_1^{*2} s_2^{*2} + A_1^2 \lambda_1^* s_2^{*2} + \lambda_2^* s_1^{*2}) s_3^{*2} s_4^{*2} \geq 0,$$

$$\begin{aligned}\det(H^{10}) = {} & 256 B_1 B_2 \prod_{i=1}^{4} s_i^{*2} + 128 B_1 \lambda_3^* s_1^{*2} s_2^{*2} s_4^{*2} + 128(B_1 + B_2)\lambda_2^* s_1^{*2} s_3^{*2} s_4^{*2} \\ & + 128(A_1^2 B_1 + A_2^2 B_1)\lambda_1^* s_2^{*2} s_3^{*2} s_4^{*2} + 64\lambda_2^* \lambda_3^* s_1^{*2} s_4^{*2} \\ & + 64 A_1^2 \lambda_1^* \lambda_3^* s_2^{*2} s_4^{*2} + 64(A_1 - A_2)^2 \lambda_1^* \lambda_2^* s_3^{*2} s_4^{*2} \\ \geq {} & 0,\end{aligned}$$

$$\begin{aligned}\det(H^{11}) = {} & 256 B_1 B_2 B_3 \prod_{i=1}^{4} s_i^{*2} + 128 B_1 B_2 \lambda_4^* s_1^{*2} s_2^{*2} s_3^{*2} + 128 B_1 (B_2 + B_3)\lambda_3^* s_1^{*2} s_2^{*2} s_4^{*2} \\ & + 128(B_1 + B_2) B_3 \lambda_2^* s_1^{*2} s_3^{*2} s_4^{*2} + 128(A_1^2 B_2 B_3 + A_2^2 B_1 B_3 + A_3^2 B_1 B_2)\lambda_1^* s_2^{*2} s_3^{*2} s_4^{*2} \\ & + 64 B_1 \lambda_3^* \lambda_4^* s_1^{*2} s_2^{*2} + 64(B_1 + B_2)\lambda_2^* \lambda_4^* s_1^{*2} s_3^{*2} + 64(A_1^2 B_2 + A_2^2 B_1)\lambda_1^* \lambda_4^* s_2^{*2} s_3^{*2} \\ & + 64(B_1 + B_2 + B_3)\lambda_2^* \lambda_3^* s_1^{*2} s_4^{*2} + 64\left[A_1^2 (B_2 + B_3) + (A_2 - A_3)^2 B_1\right]\lambda_1^* \lambda_3^* s_2^{*2} s_4^{*2} \\ & + 64\left[(A_1 - A_2)^2 B_3 + A_3^2 (B_1 + B_2)\right]\lambda_1^* \lambda_2^* s_3^{*2} s_4^{*2} + 32\lambda_2^* \lambda_3^* \lambda_4^* s_1^{*2} \\ & + 32 A_1^2 \lambda_1^* \lambda_3^* \lambda_4^* s_2^{*2} + 32(A_1 - A_2)^2 \lambda_1^* \lambda_2^* \lambda_4^* s_3^{*2} + 32(A_1 - A_2 + A_3)^2 \lambda_1^* \lambda_2^* \lambda_3^* s_4^{*2} \\ \geq {} & 0,\end{aligned}$$

$$\begin{aligned}\det(H^{12}) = {} & 256\prod_{i=1}^{4}\left(B_i s_i^{*2}\right) + 128 B_1 B_2 (B_3 + B_4)\lambda_4^* s_1^{*2} s_2^{*2} s_3^{*2} \\ & + 128 B_1 B_4 (B_2 + B_3)\lambda_3^* s_1^{*2} s_2^{*2} s_4^{*2} + 128 B_3 B_4 (B_1 + B_2)\lambda_2^* s_1^{*2} s_3^{*2} s_4^{*2} \\ & + 128(A_1^2 B_2 B_3 B_4 + A_2^2 B_1 B_3 B_4 + A_3^2 B_1 B_2 B_4 + A_4^2 B_1 B_2 B_3)\lambda_1^* s_2^{*2} s_3^{*2} s_4^{*2} \\ & + 64 B_1 (B_2 + B_3 + B_4)\lambda_3^* \lambda_4^* s_1^{*2} s_2^{*2} + 64(B_1 + B_2)(B_3 + B_4)\lambda_2^* \lambda_4^* s_1^{*2} s_3^{*2} \\ & + 64\left[(A_1^2 B_2 + A_2^2 B_1)(B_3 + B_4) + B_1 B_2 (A_3 - A_4)^2\right]\lambda_1^* \lambda_4^* s_2^{*2} s_3^{*2} \\ & + 64 B_4 (B_1 + B_2 + B_3)\lambda_2^* \lambda_3^* s_1^{*2} s_4^{*2} \\ & + 64\left[(A_1^2 B_4 + A_4^2 B_1)(B_2 + B_3) + (A_2 - A_3)^2 B_1 B_4\right]\lambda_1^* \lambda_3^* s_2^{*2} s_4^{*2} \\ & + 64\left[(A_3^2 B_4 + A_4^2 B_3)(B_1 + B_2) + (A_1 - A_2)^2 B_3 B_4\right]\lambda_1^* \lambda_2^* s_3^{*2} s_4^{*2}\end{aligned}$$

$$+32(B_1+B_2+B_3+B_4)\lambda_2^*\lambda_3^*\lambda_4^* s_1^{*2}$$
$$+32\left[A_1^2(B_2+B_3+B_4)+(A_2-A_3-A_4)^2 B_1\right]\lambda_1^*\lambda_3^*\lambda_4^* s_2^{*2}$$
$$+32\left[(A_3-A_4)^2(B_1+B_2)+(A_1-A_2)^2(B_3+B_4)\right]\lambda_1^*\lambda_2^*\lambda_4^* s_3^{*2}$$
$$+32\left[A_4^2(B_1+B_2+B_3)+(A_1-A_2+A_3)^2 B_4\right]\lambda_1^*\lambda_2^*\lambda_3^* s_4^{*2}$$
$$+16(A_1-A_2+A_3-A_4)^2\lambda_1^*\lambda_2^*\lambda_3^*\lambda_4^*$$
$$\geq 0,$$

and H is positive semidefinite.

II. Proof of formula (19)

Let ω_l and ω_u be any two cutoff values of a single-stage screening procedure. Then

$$AOQ = P(l \le Y \le u \mid \omega_1 \le X \le \omega_2)$$
$$\le \max_{x \in [\omega_1,\omega_2]} P(l \le Y \le u \mid x)$$
$$\le \max_{x \in (-\infty,\infty)} P(l \le Y \le u \mid x)$$

Since $Y|x$ is normally distributed with mean $\mu_y + \rho(\sigma_y/\sigma_x)(x-\mu_x)$ and variance $\sigma_y^2(1-\rho^2)$, $P(l \le Y \le u|x)$ can be rewritten as

$$P(l \le Y \le u|x) = \Phi\left(\frac{\tau_u - \rho z}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{\tau_l - \rho z}{\sqrt{1-\rho^2}}\right),$$

where $z = (x-\mu_x)/\sigma_x$. By differentiating this equation with respect to z, we obtain $z^* = (\tau_l + \tau_u)/(2\rho)$ and

$$\max_{x \in (-\infty,\infty)} P(l \le Y \le u|x) = 2\Phi\left((\tau_u - \tau_l)/2\sqrt{1-\rho^2}\right) - 1.$$

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95년 7월 최초 접수, 95년 12월 최종 수정