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A (Q, r) Spare-Part Inventory Model with Gamma Leadtime

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Abstract

This paper deals with a (Q, r) spare-part inventory model with gamma leadtime. In the model, if the inventory level falls to a reorder point r, a replenishment order quantity Q is ordered. Assumming that the number of operating units is one and the lifetime of a unit follows an exponential distribution, we derive the expected cost rate and suggest a procedure to obtain the optimal pair of (Q, r) minimizing the cost rate. A numerical example is presented to explain the model.

1. Introduction

A spare-part inventory model differs from the classical inventory problem in that the demand for the part never arises during stockout period, since the unit under consideration remains inoperative when stockout occurs until the failed part is replaced by new one. Karlin(1958) introduced a (Q, r) spare-part inventory model with exponential lifetime and leadtime distributions and derived some probability quantities associated with the inventory model, but his results contain some errors. Park(1986) corrected Karlin's results and developed an optimization algorithm for the problem, and Park et. al.(1995) extended it to gamma lifetime case. Falkner(1969) treated a similar problem, but his study was a single-period model in which the procurement of spare units was allowed once only at the beginning of planning horizon, and the problem was to determine the initial stock level so as to minimize the expected total cost. Park(1981) considered a one-for-one ordering spare-part inventory model for fleet maintenance, where a number of units are operating simultaneously. Since the one-for-one ordering model is a particular case of (Q, r) model with $Q=1$, the problem was to determine the optimal inventory position. This paper extends Karlin's(1958) and Park's(1986) model to gamma leadtime case.

2. Assumptions and Notations

Assumptions

1. The unit cost of a unit is a constant regardless of the order quantity.
2. The distribution of service life of a unit is exponential.
3. The distribution of a replenishment leadtime is gamma.
4. There is never more than a single order outstanding.

Notations

Q, r	order quantity per order and reorder point, respectively,
λ	failure rate of a unit,
$g(y;\, q,\, \mu)$	$\frac{\mu(\mu y)^{q-1} exp(-\mu y)}{\Gamma(q)}$, gamma p.d.f. with parameters q and μ denoting a replenishment leadtime,
$p(j;\, m)$	$\frac{m^j e^{-m}}{j!}$, Poisson p.m.f. with parameter m,
$b_N(j;\, q,\, \theta)$	$\frac{\Gamma(j+q)}{j!\Gamma(q)} \theta^q (1-\theta)^j$, negative binomial p.m.f. with parameters q and θ,
c_o	fixed ordering cost per order,
c_h	holding cost rate of a unit,
c_s	shortage cost rate of having a unit idle.

Other notations are defined as needed.

3. Cost Rate Derivation

In this model, the stock level (that is, the number of spare units) is reviewed continuously, and an order for quantity Q is placed when the stock level drops to a reorder point r. Imbedded in the implementation of the (Q, r) policy is a natural cycle which commences when an order is delivered and lasts until the next order is delivered. The special nature of the exponential lifetime distribution, which in effect implies that any conditional density is independent of the conditional statement enables us to start the process anew when delivery takes place, regardless of the age of the unit in use [Karlin, 1958].

The expected cost per cycle is the sum of the ordering, holding and shortage costs. Since the number of orders per cycle is one, the ordering cost per cycle is c_o. The expected holding cost per cycle will be computed in two parts. First, for the time period between an order arrival and the next order, and second for the replenishment leadtime y between an order and the next arrival. Given that v units are on hand just after an order arrives, the expected unit years in stock held until the reorder point is reached is :

$$\frac{v+(v-1)+(v-2)+\cdots\cdots+(r+1)}{\lambda}=\frac{v(v+1)-r(r-1)}{2\lambda} \tag{1}$$

The probability that v units are on hand just after the arrival of an order is :

$$\pi(v)=\begin{cases}1-\sum_{j=0}^{r}\int_0^{\infty}p(j;\lambda y)\,g(y;q,\mu)\,dy=1-\sum_{j=0}^{r}b_N(j;q,\theta), & \text{if } v=Q-1\\ \int_0^{\infty}p(Q+r-v;\lambda y)\,g(y;q,\mu)\,dy=b_N(Q+r-v;q,\theta), & \text{if } Q\le v\le Q+r\end{cases} \tag{2}$$

where, $\theta=\mu/(\lambda+\mu)$.

Averaging Equation (1) over the initial inventory v, the expected unit years stock held until the reorder point is reached is :

$$\begin{aligned}&\sum_{v=Q}^{Q+r}\pi(v)\left[\frac{v(v+1)-r(r+1)}{2\lambda}\right]\\ &=\frac{1}{2\lambda}\left[Q(Q-1)-r(r+1)\right]\left[1-\sum_{j=0}^{r}b_N(j;q,\theta)\right]\\ &\quad+\frac{1}{2\lambda}\sum_{v=Q}^{Q+r}\left[\{v(v+1)-r(r+1)\}\,b_N(Q+r-v;q,\theta)\right]\\ &=\frac{1}{2\lambda}\left[Q(Q-1)-r(r+1)\right]\\ &\quad+\frac{1}{2\lambda}\sum_{j=0}^{r}(2Q+r-j)(r-j+1)\,b_N(j;q,\theta)\end{aligned} \tag{3}$$

The expected unit years of spares in stock held during the time period from a reorder to the next arrival is the integral during the replenishment leadtime y of the expected amount of spares on hand, i.e.,

$$\begin{aligned}&\int_0^{\infty}\int_0^{y}\sum_{j=0}^{r}(r-j)\,p(j;\lambda t)\,g(y;q,\mu)\,dt\,dy\\ &=\frac{r(r+1)}{2\lambda}-\frac{1}{\lambda}\sum_{j=0}^{r}(r-j)\sum_{i=0}^{j}b_N(i;q,\theta)\\ &=\frac{1}{2\lambda}\left[r(r+1)-\sum_{j=0}^{r}(r-j)(r-j+1)\,b_N(j;q,\theta)\right]\end{aligned} \tag{4}$$

On summing Equations (3) and (4), and making some algebraic manipulations, we find that the expected number of unit years of stock held per cycle is :

$$\frac{Q}{\lambda}\left[\frac{Q-1}{2}+A(r)\right] \tag{5}$$

where, $A(r) = \sum_{j=0}^{r}(r-j+1)\, b_N(j;\, q,\, \theta)$

Let us now compute the expected shortage cost per cycle. If the system reaches a stockout condition in the time interval between t and $t+dt$ after a reorder point r is hit, this implies that in the time 0 to t, r units have been demanded and the $(r+1)$th one is demanded between t and $t+dt$. Hence it will be stockout for a length of time $y-t$ during the cycle, and the expected length of stockout per cycle is :

$$\int_0^{\infty}\int_0^{y}\lambda(y-t)\, p(r;\, \lambda t)\, g(y;\, q,\, \mu)\, dt\, dy$$

$$= \frac{q}{\mu}-\frac{r+1}{\lambda}+\frac{1}{\lambda}A(r) \tag{6}$$

From Equations (5) and (6), the expected total cost per cycle is :

$$c_o+\frac{c_h Q}{\lambda}\left[\frac{Q-1}{2}+A(r)\right]+c_s\left[\frac{q}{\mu}-\frac{r+1}{\lambda}+\frac{1}{\lambda}A(r)\right] \tag{7}$$

Since the time between successive replenishments is a cycle, the expected duration of a cycle is the expected time between the arrival of an order and the next order placement plus the expected replenishment leadtime. Thus expected cycle length is :

$$\sum_{v=Q-1}^{Q+r}\pi(v)\,\frac{(v-r)}{\lambda}+\int_0^{\infty}y\, g(y;\, q,\, \mu)\, dy$$

$$= \frac{Q-1-r}{\lambda}\left[1-\sum_{j=0}^{r}b_N(j;\, q,\, \theta)\right]+\sum_{v=Q}^{Q+r}\frac{(v-r)}{\lambda}b_N(Q+r-v;\, q,\, \theta)+\frac{q}{\mu}$$

$$= \frac{Q}{\lambda}+\frac{q}{\mu}-\frac{r+1}{\lambda}+\frac{1}{\lambda}A(r) \tag{8}$$

The expected cost rate for an infinite time span is the expected cost per cycle divided by the expected cycle length. Hence, the expected cost rate is :

$$C(Q, r) = \frac{c_o + \frac{c_h Q}{\lambda}\left[\frac{Q-1}{2} + A(r)\right] + c_s\left[\frac{q}{\mu} - \frac{r+1}{\lambda} + \frac{1}{\lambda}A(r)\right]}{\frac{Q}{\lambda} + \frac{q}{\mu} - \frac{r+1}{\lambda} + \frac{1}{\lambda}A(r)} \tag{9}$$

where, $A(r) = \sum_{j=0}^{r}(r-j+1)\, b_N(j;\, q,\, \theta)$ and $\theta = \mu/(\lambda+\mu)$.

If the leadtime distribution is exponential (i.e., $q=1$), then the expected cost rate can be reduced as follows :

$$C(Q, r) = \frac{c_o + \frac{c_h Q}{\lambda}\left[\frac{Q-1}{2} + (r+1) - \frac{\lambda}{\mu} + \frac{\lambda}{\mu}\rho^{r+1}\right] + \frac{c_s}{\mu}\rho^{r+1}}{\frac{Q}{\lambda} + \frac{1}{\mu}\rho^{r+1}} \tag{10}$$

where, $\rho = 1-\theta = \lambda/(\lambda+\mu)$

Equation (10) coincides with the result of Park's(1986).

4. The Iterative Optimization Procedure and Numerical Example

The optimal replenishment order quantity Q^* and the optimal reorder point r^* can be obtained by using the following iterative optimization procedure as in Hadley and Whitin's(1963) :

Step 1 : The initial estimate for Q is $(2\lambda c_o/c_h)^{1/2}$ (; Wilson's lot size formula). Call this value Q_1.

Step 2 : Given that the value of $Q=Q_1$, find the value of r minimizing $C(Q, r)$. Call this value r_1.

Step 3 : Given that the value of $r=r_1$, find the value of Q minimizing $C(Q, r)$. Call this value Q_2.

Step 4 : Repeat Step 2 with $Q=Q_2$ etc. Convergence occurs when at iteration i, $Q_i = Q_{i-1}$ or $r_i = r_{i-1}$.

In order to explain the iterative procedure, consider the following case :

$c_o = \$100,\ c_h = \$5,\ c_s = \$1000,\ \lambda = 2,\ g(y;\, q,\, \mu) = g(y;\, 3,\, 1)$.

The iterative optimization procedures are as follows :

Step 1 : $Q_1 = (2\lambda c_o / c_h)^{1/2} = (2 \times 2 \times \$100/\$5)^{1/2} \doteqdot 9$

Step 2 : $r_1 = 8$, $C(Q_1, r_1) = \$131.18$

Step 3 : $Q_2 = 18$, $C(Q_2, r_1) = \$108.33$

Step 4 : $r_2 = 11$, $C(Q_2, r_2) = \$100.28$

Step 3 : $Q_3 = 14$, $C(Q_3, r_2) = \$97.83$

Step 4 : $r_3 = 12$, $C(Q_3, r_3) = \$97.33$

Step 3 : $Q_4 = 13$, $C(Q_4, r_3) = \$97.05$

Step 4 : $r_4 = 12$, Since $r_4 = r_3$, the iteration stops here.

$$Q^* = Q_4 = 13, \quad r^* = r_4 = 12, \quad C(Q^*, r^*) = \$97.05$$

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