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# An Anderson-Darling Goodness-of-Fit Test for the Gamma Distribution

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## Abstract

This paper provides a test of the composite hypothesis that a random sample is (two parameter) gamma distributed when both the scale and shape parameters are estimated from the data. The test statistic is a variant of the usual Anderson-Darling statistic, the primary difference being that the statistic is based on the maximum likelihood estimator of the shape parameter of the assumed gamma distribution. The percentage points are developed via simulation and are presented graphically. Examples are provided.

## 1. Introduction

The gamma family of distributions is widely used in life and reliability studies, in large measure because of its distinctive failure rate function. It is therefore of some interest that a test be available to determine whether in the course of modeling of or analysis of lifetime data the assumption of a gamma distribution is warranted. We are primarily concerned with the two-parameter gamma distribution although our results can be extended to several more complicated settings involving the gamma distribution, the three parameter gamma for example. Our results cover only the case in which both scale and shape parameters are unknown and must be estimated from the data; other cases in which combinations of parameters are known must be handled separately. We believe, however, that the case we consider is the most useful in a life testing and reliability context. The goodness of fit problem in which the parameters are unknown and must be estimated from the data is most easily approached by one of the variants of the chi-squared procedure (Moore and Stubblebine, 1981).

However, the chi-squared approach is likely not to be as powerful as procedures based on empirical distribution functions, Anderson-Darling procedures for example. Relevant work for testing for the gamma distribution has been done by Lillifors (1969), by Durbin (1975), and especially Woodruff, Viviano, Moore, and Dunne (1984).

This paper presents in a graphical format percentage points of modified Anderson-Darling statistic $\widehat{A}_n^2$ for a test of the two-parameter gamma distribution when both parameters are estimated from the data for sample size $n$ = 10, 15, 20, 24, 30, 40, 50, 60, 120, for shape parameter less than 20, and for size of test 0.10, 0.05, 0.025, and 0.01.

## 2. Problem Statement

Notation

| | |
|---|---|
| $n$ | random sample size |
| $A_n^2$ | Anderson-Darling statistic |
| $\widehat{A}_n^2$ | Anderson-Darling type statistic with maximum likelihood estimates |
| $\alpha$ | shape parameter of gamma distribution |
| $\beta$ | scale parameter of gamma distribution |
| $\widehat{\alpha}$ | maximum likelihood estimate of $\alpha$ |
| $\widehat{\beta}$ | maximum likelihood estimate of $\beta$ |
| $f(x: \alpha, \beta)$ | gamma density with shape parameter $\alpha$, scale parameter $\beta$ |
| $g(x: \alpha, \phi)$ | log-gamma density with shape parameter $\alpha$, location parameter $\phi = \log \beta$ |

Assumptions

1. $x_1, x_2, \cdots, x_n$ is a complete random sample from an unknown distribution, but putatively $f(x: \alpha, \beta)$ .
2. The $x_i$ are statistically independent.
3. The shape parameter $\alpha$ and the scale parameter $\beta$ are not specified in the hypothesized null distribution.

The problem addressed here is to determine whether a random sample $x_1, x_2, \cdots, x_n$ comes from an unspecified two-parameter gamma distribution.

# 3. The Anderson-Darling Statistic

Suppose, on the null hypothesis, that $x_1, x_2, \cdots, x_n$ is a random sample drawn from density

$$f(x: \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta} [x/\beta]^{\alpha-1} \exp[-x/\beta], \quad x \geq 0, \quad \alpha > 0, \quad \beta > 0 \tag{1}$$

where $\Gamma(\alpha)$ is the gamma function with argument $\alpha$. The unknown parameters $\alpha$ and $\beta$ are estimated from the data by determining the values of $\alpha$ and $\beta$, $\hat{\alpha}$ and $\hat{\beta}$, which maximize the log likelihood

$$1_0(\alpha, \beta) = \sum_{i=1}^{n} \log f(x_i: \alpha, \beta) \tag{2}$$

Maximization of $1_0(\alpha, \beta)$ with respect to α and $\beta$ provides estimators $\hat{\alpha}$ and $\hat{\beta}$ for the unknown shape parameter $\alpha$ and the unknown scale parameter $\beta$. Some care must be exercised in determining $\hat{\alpha}$ and $\hat{\beta}$ when $\alpha < 1$ . However, several alternatives to direct use of (1) are generally more useful in practical applications. One alternative is to make the transformation $y = \log x$ in (1) to get the density

$$g(y: \alpha, \phi) = \frac{1}{\Gamma(\alpha)} \exp[\alpha(y-\phi) - \exp(y-\phi)] \tag{3}$$

where

$$\phi = \log \beta,$$

and to maximize $1_0(\alpha, \phi) = \sum \log g(y_i: \alpha, \phi)$ to determine the maximum likelihood estimators for α and $\beta$. A second alternative is to maximize $1_0(\alpha, \phi)$ subject to the constraint that the distribution function $G(y: \alpha, \phi)$ corresponding to $g(y: \alpha, \phi)$ satisfies

$$G(y_{(i)}: \alpha, \phi) \geq 1/(n+1) \tag{4}$$

where $y_{(1)}, y_{(2)}, \cdots, y_{(n)}$ are the $y_j$'s arranged in ascending order. This second alternative is also often very useful in working with (1).

For completely specified $f(x: \alpha, \beta)$ of gamma variates $x_1, x_2, \cdots, x_n$, the Anderson-Darling statistic $A_n^2$ is

$$A_n^2 = -\sum_{i=1}^{n} \frac{2i-1}{n} [\log F(x_{(i)}) + \log[1-F(x_{(n+1-i)})]] - n, \quad (5)$$

where

$$F(x) = \int_0^x f(x': \alpha, \beta)\, dx'$$

and the $x_{(i)}$ are the $x_i$ placed in ascending order; i.e., the order statistics(Lawless, 1982). However, we are exclusively concerned with the case in which $\alpha$ and $\beta$ are unknown and must be estimated from the data. We define $\widehat{A}_n^2$ to be the statistic obtained by replacing $\alpha$ and $\beta$ by $\widehat{\alpha}$ and $\widehat{\beta}$ respectively. In order to test the composite hypothesis that $x_1, x_2, \cdots, x_n$, is a random sample of failure times from the gamma distribution via $\widehat{A}_n^2$, we need the distribution of $\widehat{A}_n^2$ or given values of $n$. The distribution of $\widehat{A}_n^2$ also depends on the unknown value of the shape parameter $\alpha$. We observe that the result $\widehat{\alpha}$ of the process of maximum likelihood estimation could have been generated by a continuum of unknown $\alpha$ values and this provides the key to constructing an Anderson-Darling test for the gamma distribution. We shall construct via an extensive simulation, a test in such a way that the percentage points of $\widehat{A}_n^2$ are indexed on the value $\widehat{\alpha}$. The following illustration is representative of the general construction and is the basis of the construction of a test statistic.

Suppose we simulate 1000 random sample of $n$=20 gamma variates with shape parameters $\alpha$=0.25 and scale parameters $\beta$=1, 1000 random samples of $n$=20 gamma variates with $\alpha$=0.35, $\beta$=1, 1000 random samples of $n$=20 gamma variates with $\alpha$=0.45, $\beta$=1, ..., 1000 gamma variates with $\alpha$=1.75, $\beta$=1. All the variates are mutually independent. For each random sample determine the triple $(\widehat{\alpha}, \widehat{\beta}, \widehat{A}_n^2)$. Sort the pairs $(\widehat{\alpha}, \widehat{A}_n^2)$ in accordance with $\widehat{\alpha} \le 0.25$, $0.25 < \widehat{\alpha} \le 0.35$, ..., $0.95 < \widehat{\alpha} \le 1.05$, ..., $\widehat{\alpha} > 1.75$. For those pairs $(\widehat{\alpha}, \widehat{A}_n^2)$ with $0.95 < \widehat{\alpha} \le 1.05$, arrange the $\widehat{A}_n^2$ in ascending order. In this interval centered on $\widehat{\alpha}$ = 1, there will be a random number of $\widehat{A}_n^2$ values, $N(1)$, say. From the order statistics in $\widehat{A}_n^2$.

we determine the 90, 95, 97.5 and 99 percent points of the distribution of $\widehat{A}_n^2$ at $\widehat{\alpha} = 1$. It is necessary to simulate from gamma distribution with $\alpha$ values far away from $\alpha=1$ in order to avoid "end effect", i.e., to allow every gamma distribution which can contribute a realization $0.95 < \hat{\alpha} \leq 1.05$ to do so. Clearly to restrict α to $0.95 < \alpha \leq 1.05$ would not be sufficient.

We carried out this strategy on a much more extensive scale, simulating gamma variates with $\beta=1$ in independent samples of size $n$ for $\alpha$ values $\alpha= 0.05(0.10)$ 25.05, $n$=10, 15, 20, 24, 30, 40, 60, 120. For each sample generated, we computed $\hat{\alpha}$, $\hat{\beta}$, $\widehat{A}_n^2$ through (3) and (5) and sorted the pairs $(\hat{\alpha}, \widehat{A}_n^2)$ according to $0 < \hat{\alpha} \leq 0.15$, $0.15 < \hat{\alpha} \leq 0.25$, .... Within each of these intervals the 90, 95, 97.5 and 99 percent points of $\widehat{A}_n^2$, $n$ fixed, were computed and plotted against the midpoint of that interval. These discrete relationships in the 90, 95, 97.5, and 99 percent point against $\widehat{\alpha}$ were found to be very well-behaved. Continuous functions of $\widehat{\alpha}$ ere then produced by fitting spline functions to the discrete points. Remarkably, the resulting graphical representation of the percentage points for testing for the composite hypothesis of gamma distribution were virtually coincident for all sample sizes $n \geq 10$. These graphical representations also showed that some additional sampling was needed to clarify the values of percentage points for a near zero. Additional simulation were used to augment the original simulation study. Finally, <Figure 1> provides the percentage points of the statistic $\widehat{A}_n^2$ for the composite test for gamma distribution for all sample sizes $n \geq 10$.

# 4. A Limited Power Study And An Example

We now examine the power performance of the statistic $\widehat{A}_n^2$ under alternative life distributions for the sample size $n$=20 and 60. The following distributions were considered.

1. Log-normal $(\mu, \sigma)$

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-(\log x-\mu)^2/(2\alpha^2)\right], \quad x>0 \tag{6}$$

2. Weibull $(\alpha, \beta)$

$$f(x) = \alpha\beta^{-\alpha}x^{\alpha-1}\exp[-(x/\beta)^{\alpha}],\ x>0,\ \alpha,\beta>0 \tag{7}$$

3. Uniform $(a, b)$

$$f(x) = 1/(b-a), a\le x\le b \tag{8}$$

4. Beta $(p, q)$

$$f(x) = (1/B(p,q))x^{p-1}(1-x)^{q-1}, 0<x<1,\ p,q>0, \tag{9}$$

where $B(p, q) = \Gamma(p+q+1)/(\Gamma(p)\Gamma(q))$.

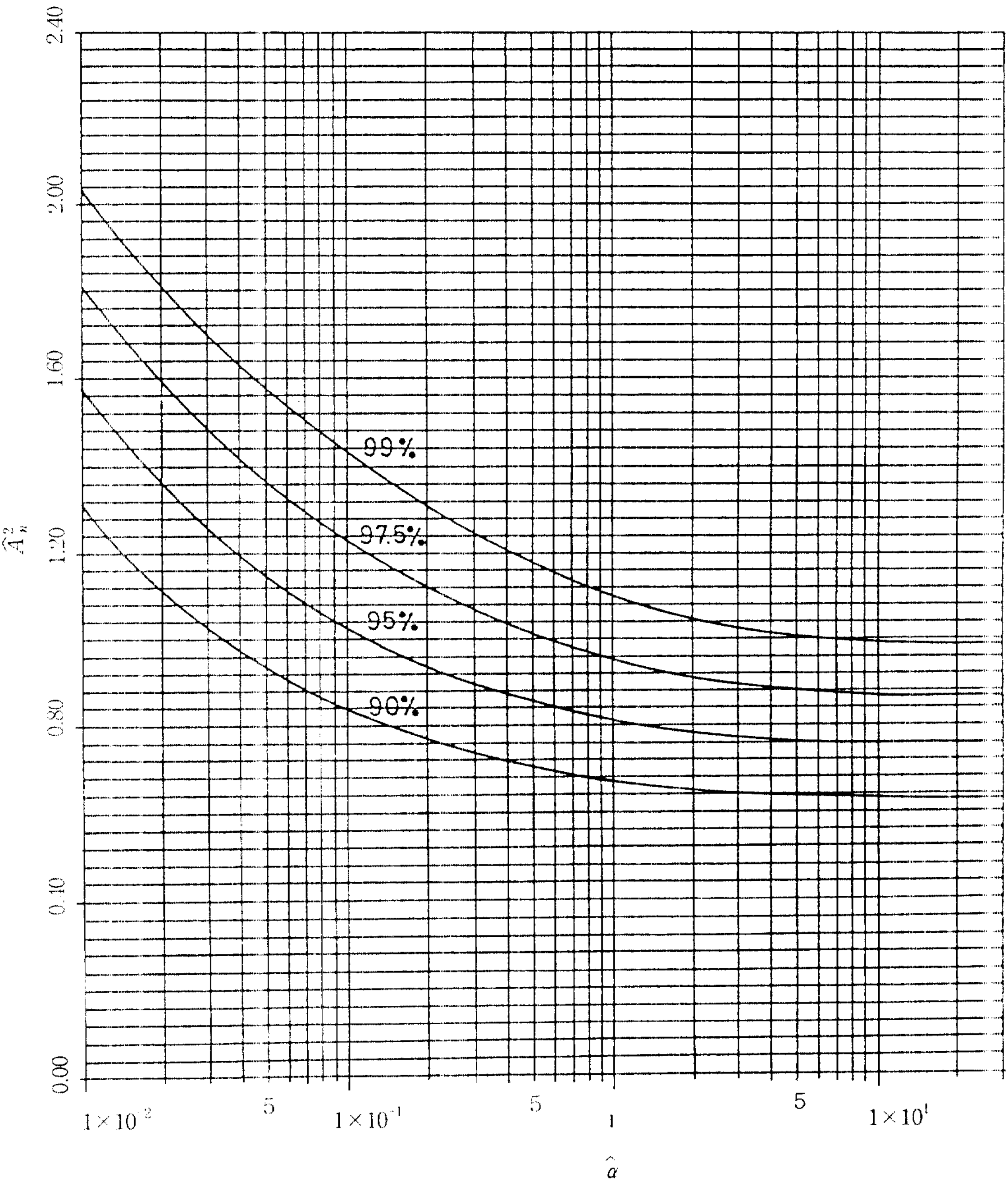
5. Extreme Value $(b)$

$$f(x) = (1/b)\exp[x/b-\exp(x/b)],\ -\infty<x<\infty,\ b>0 \tag{10}$$

Since $x$ is defined in $-\infty<x<\infty$, the variates of extreme value are shifted to the positive side by 5 times the sample standard deviation of $x$.

We generated 1000 random samples from each of the alternative distributions and calculated the ratios of cases that the value of the statistic $\widehat{A}_n^2$ at $\widehat{\alpha}$ has its position above the 95% contour line. <Table 1> shows the corresponding powers for the significance level of 0.05.

Proschan discusses the life distribution of air conditioners installed in various fleets of aircraft (1963). He suggests that the total sample of 213 lifetimes does not follow an exponential life distribution because these lifetimes are characterized by a decreasing failure rate. Dahiya and Gurland (1972) suggest, based on their moment-based statistic of fit, that these lifetimes are consistent with a two-parameter gamma life distribution of (1). Our results indicate that the two-parameter gamma lifetime for these data is not warranted. We find the maximum likelihood estimates $\widehat{\alpha}$ = 0.922 with $\widehat{A}_n^2$ = 1.124. Comparing $\widehat{A}_n^2$ = 1.124 with the 95 % contour in <Figure 1> at $\widehat{\alpha}$ = 0.922, we clearly reject the null gammaness of the air conditions data. It is positioned just above the 99% contour line.




< Figure 1 > Critical contour lines of $\widehat{A}_n{}^2$

< Table 1 > Power study of $\hat{A}_n^2$

| alternative distributions | | sample size 20 | 60 |
|---|---|---|---|
| Log-normal | (0, 0.25) | 0.058 | 0.086 |
| | (0, 0.50) | 0.081 | 0.185 |
| | (0, 1,50) | 0.375 | 0.846 |
| | (0, 3.00) | 0.630 | 0.988 |
| Weibull | (3.0, 1.0) | 0.139 | 0.308 |
| | (1.5, 1.0) | 0.057 | 0.089 |
| | (1.0, 1.0) | 0.045 | 0.048 |
| | (0.5, 1.0) | 0.106 | 0.277 |
| Uniform | (0.0, 1.0) | 0.440 | 0.962 |
| Beta | (0.5, 1.0) | 0.258 | 0.761 |
| | (0.5, 2.0) | 0.090 | 0.221 |
| | (1.0, 2.0) | 0.122 | 0.479 |
| | (2.0, 4.0) | 0.078 | 0.219 |
| Extreme Value | (0.5) | 0.158 | 0.359 |
| (shifted) | (1.0) | 0.140 | 0.357 |
| | (3.0) | 0.134 | 0.350 |
| | (5.0) | 0.150 | 0.344 |

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