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Optimal Plan for Fully Accelerated Life Tests with Three-Step Stress Under Type I Censoring

Kyoung Ae Moon and Im Hee Shin

Abstract In this paper, optimal change times are determined for fully three-step stress accelerated life tests, which minimize the asymptotic variance for maximum likelihood estimator of logarithm of the failure rate at the usual condition and exponential distribution is given for life time data.

1. Introduction

The situation where all three stress settings s_1,s_2 and s_3 of the step stress tests are larger than the use stress s_u, which is referred to as fully accelerated life tests(ALTs), is considered in this paper. In general, fully ALTs is more appropriate than partially ALTs, in which the initial stress is the same as the use stress s_u, if life tests at the usual condition require too much time. However, an acceleration function on the stress variable must be specified under fully ALTs, which relates to the parameters under the step stress tests with those at the usual condition.

First of all, an acceleration function on the stress variable to describe the relationship between parameters under three-step stress tests and those at the usual condition is introduced, and then maximum likelihood estimators(MLEs) of parameters are obtained. The problem of plan which determinines optimal stress change times under the step stress ALTs seems to be important for improving the precision in estimating relationships between lifetime and stress. Thus optimal change times are searched for three-step stress ALTs. At that time, the asymptotic variance of MLE of logarithm of the failure rate at the usual condition is considered as the criterion for optimality. In practice, to use this optimal plan, the unknown parameters in optimal change times must be approximated by experience, or preliminary tests.

Chernoff(1962) calls such a plan *locally optimum*. However the value that are approximated erroneously may make a plan far from optimum. This possibility can be checked if the relative increase of the asymptotic variance of MLE of logarithm of the failure rate at the usual condition when the wrong values are used for searching optimal stress change times compared to the optimal asymptotic variance are examined.

Kyoung Ae Moon is instructor of Donghae Junior College, Department of Office Automation and Im Hee Shin is lecturer of Kyungpook National University, Department of Statistics.

2. Models and Maximum Likelihood Estimators

For fully three-step stress ALTs, all units are simultaneously put on the initial stress s_1 until a preassigned time τ_1, but if all units do not fail before time τ_1, the surviving units are subjected to the increased stress s_2 and observed until time τ_2, The rest units functioning at time τ_2 are also subjected to a larger stress s_3 and the stress helds constant until the preassigned censoring time T. By changing the stress at the preassigned time, the failure rate function at the stress s_2 is assumed to be expressed as the initial failure rate function λ_1 at the stress s_1 multiplied by an unknown factor α_1, and the failure rate function at the stress s_3 as λ_1 multiplied by an unknown factor $\alpha_1\alpha_2$. Such a failure rate function is referred to as the tampered failure rate(TFR) function of the random variable X introduced by Bhattacharrya and Soejoeti(1989), which is defined as

$$\lambda(x) = \begin{cases} \lambda_1 & \text{if} \quad x \le \tau_1 \\ \alpha_1\lambda_1 & \text{if} \quad \tau_1 < x \le \tau_2 \\ \alpha_1\alpha_2\lambda_1 & \text{if} \quad \tau_2 < x \end{cases} \tag{1}$$

Acceleration factors α_i, $i = 1,2$ depend on the stresses s_1, s_2 and s_3 and possibly τ_1 and τ_2. Acceleration factors α_1 and α_2 will also be larger than 1 because the test unit is subjected to a greater failure in changing the stress to the higher level.

It is assumed that the failure rate function of each test unit has the log-quadratic relationship with the stress variable s, which is given by

$$\log \lambda(x) = -\nu_1 - \nu_2 s - \nu_3 s^2 \tag{2}$$

where ν_1, ν_2 and ν_3 denote the regression parameters.

Assume that each test unit follows an exponential distribution with failure rate λ_1. Then for the TFR model under the three-step stress ALTs, the probability density function of the lifetime X is given by

$$f(x) = \begin{cases} \lambda_1 \exp(-\lambda_1 x) & \text{if} \quad x \le \tau_1 \\ \alpha_1\lambda_1 \exp(-\alpha_1\lambda_1(x-\tau_1) - \lambda_1\tau_1) & \text{if} \quad \tau_1 < x \le \tau_2 \\ \alpha_1\alpha_2\lambda_1 \exp(-\alpha_1\alpha_2\lambda_1(x-\tau_2) & \text{if} \quad \tau_2 < x \\ \quad -\alpha_1\lambda_1(\tau_2-\tau_1) - \lambda_1\tau_1) & \end{cases} \tag{3}$$

Let $\lambda_2 = \alpha_1\lambda_1$ and $\lambda_3 = \alpha_1\alpha_2\lambda_1$. From these notations and equation (2), it can be seen that model parameters λ_1, λ_2 and λ_3 are related to regression parameters ν_1, ν_2 and ν_3 via the following equations

$$\begin{aligned}\log\lambda_1 &= -v_1 - v_2 s_1 - v_3 s_1^2\\ \log\lambda_2 &= -v_1 - v_2 s_2 - v_3 s_2^2\\ \log\lambda_3 &= -v_1 - v_2 s_3 - v_3 s_3^2\end{aligned} \tag{4}$$

The failure rate of the lifetime X at the usual condition is then given by

$$\log\lambda_u = -v_1 - v_2 s_u - v_3 s_u^2 \tag{5}$$

where s_u and λ_u mean the use stress and the failure rate at the usual condition, respectively, and our object is to estimate $\log\lambda_u$.

The MLE of the failure rate at the usual condition can be obtained by using MLEs of regression parameters of equation (5).

From equation (4), we can see that there is a one-to-one relation between original model parameters and new regression parameters. After estimating MLEs of parameters λ_1, λ_2 and λ_3 and by using them, MLEs of regression parameters can be obtained by the invariance property of the MLE.

From the results of equation (5), the MLE of the failure rate at the usual condition is given by

$$\begin{aligned}\log\hat{\lambda}_u &= -\hat{v}_1 - \hat{v}_2 s_u - \hat{v}_3 s_u^2\\ &= -d_1\log\hat{\lambda}_1 + d_2\log\hat{\lambda}_2 - d_3\log\hat{\lambda}_3\end{aligned} \tag{6}$$

where

$$\hat{\lambda}_1 = \frac{n_1}{\sum_{p_1=1}^{n_1} x_{p_1} + (n_2+n_3+n_c)\tau_1},$$

$$\hat{\lambda}_2 = \frac{n_2}{\sum_{p_2=1}^{n_2} (x_{p_2}-\tau_2) + (n_3+n_c)(\tau_2-\tau_1)},$$

$$\hat{\lambda}_3 = \frac{n_3}{\sum_{p_3=1}^{n_3} (x_{p_3}-\tau_2) + n_c(T-\tau_2)}$$

and

$$d_1 = \frac{(s_3-s_u)(s_2-s_u)}{(s_2-s_1)(s_3-s_1)},\quad d_2 = \frac{(s_3-s_u)(s_1-s_u)}{(s_2-s_1)(s_3-s_2)},\quad d_3 = \frac{(s_2-s_u)(s_1-s_u)}{(s_3-s_1)(s_3-s_2)}.$$

3. Optimal Plan for Fully ALTs

In this section, optimal stress change times under fully three-step stress ALTs are presented and the criterion for optimality is to minimize the asymptotic variance of MLE of logarithm of the failure rate at the usual condition.

Miller and Nelson(1983) studied the optimal plans for the change point that minimizes the variance of the estimator of the mean life at the usual condition, assuming an exponential distribution with a mean that is a log-linear function of stress and the complete data under simple step stress(two-step stress) ALTs. Their results were extended to the case of Type-I censoring by Bai, Kim and Lee (1989).

Let I be the Fisher information matrix about λ_1, λ_2 and λ_3 and it is then given by

$$I = n\begin{pmatrix} \frac{1-a_1}{\lambda_1^2} & 0 & 0 \\ 0 & \frac{a_1-a_2}{\lambda_2^2} & 0 \\ 0 & 0 & \frac{a_2-a_3}{\lambda_3^2} \end{pmatrix} \tag{7}$$

where

$$a_1 = \exp(-\lambda_1\tau_1),$$
$$a_2 = \exp(-\lambda_2(\tau_2-\tau_1)-\lambda_1\tau_1),$$
$$a_3 = \exp(\lambda_3(T-\tau_2)-\lambda_2(\tau_2-\tau_1)-\lambda_1\tau_1).$$

For three-step stress ALTs, the asymptotic variance of $\log\hat{\lambda}_u$ using the Fisher information matrix in equation (7) is given by

$$\sigma^2 = nVar(\log\hat{\lambda}_u) = \frac{d_1^2}{1-a_1} + \frac{d_2^2}{a_1-a_2} + \frac{d_3^2}{a_2-a_3} \tag{8}$$

From next theorem, optimal stress change times which minimize the asymptotic variance of $\log\hat{\lambda}_u$ can be obtained.

Theorem 3.1. Optimal change times τ_i^*, $i=1,2$ for three-step stress ALTs are the unique solutions of

$$\begin{aligned} \psi_1^2 &= \frac{(1-a_1)^2}{(a_1-a_2)^2 a_1}\left(a_1-a_2+\frac{\lambda_2}{\lambda_1}a_2\right)\psi_2^2 - \frac{(\lambda_2-\lambda_1)(1-a_1)^2}{(a_2-a_3)a_1\lambda_1} \\ \psi_2^2 &= \frac{(a_1-a_2)^2}{(a_2-a_3)^2 a_2}\left(a_2-a_3+\frac{\lambda_3}{\lambda_2}a_3\right) \end{aligned} \tag{9}$$

where $\psi_1 = d_1/d_3$, $\psi_2 = d_2/d_3$.

Proof. To show that σ^2 has the unique minimum value at τ_1^* and τ_2^*, it is sufficient to prove that the second order partial derivatives matrix is positive definite. By equating the first derivatives of σ^2 with respect to τ_1 and τ_2 to zero, equation (9) can be obtained, which completes the proof.

Let $p_i = 1-\exp(-\lambda_i T)$, $i = 1,2,3$ and $\zeta_1 = \tau_1/T$ and $\zeta_2 = \tau_2/T$. Then a_1, a_2 and a_3 can be expressed as

$$a_1 = (1-p_1)^{\zeta_1},$$
$$a_2 = (1-p_1)^{\zeta_1}(1-p_2)^{\zeta_2-\zeta_1},$$
$$a_3 = (1-p_1)^{\zeta_1}(1-p_2)^{\zeta_2-\zeta_1}(1-p_3)^{1-\zeta_2}.$$

Hence equation (9) can be rewritten as for censoring time T,

$$\psi_1^2 = \frac{(1-(1-p_1)^{\zeta_1})^2\psi_2^2}{(1-p_1)^{2\zeta_1}(1-(1-p_2)^{\zeta_2-\zeta_1})} + \frac{(1-(1-p_1)^{\zeta_1})^2(1-p_2)^{\zeta_2-\zeta_1}\psi_2^2}{(1-p_1)^{2\zeta_1}(1-(1-p_2)^{\zeta_2-\zeta_1})^2}\frac{\log(1-p_2)}{\log(1-p_1)}$$
$$- \frac{(1-(1-p_1)^{\zeta_1})^2}{(1-p_1)^{2\zeta_1}(1-p_2)^{\zeta_2-\zeta_1}(1-(1-p_3)^{1-\zeta_2})}\left(\frac{\log(1-p_2)}{\log(1-p_1)} - 1\right),$$
$$\psi_2^2 = \frac{(1-(1-p_2)^{\zeta_2-\zeta_1})^2}{(1-p_2)^{2(\zeta_2-\zeta_1)}(1-(1-p_3)^{1-\zeta_2})} + \frac{(1-p_3)^{1-\zeta_2}(1-(1-p_2)^{\zeta_2-\zeta_1})^2}{(1-p_2)^{2(\zeta_2-\zeta_1)}(1-(1-p_3)^{1-\zeta_2})^2}\frac{\log(1-p_3)}{\log(1-p_2)} \quad (10)$$

Therefore, optimal change times are $\tau_i^* = \zeta_i^* T$, $i = 1,2$ where ζ_i^*, $i = 1,2$ are unique solutions of equation (10) and they can be obtained by some numerical method.

References

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