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Limit of the Ratio of Incomplete Beta Functions

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Abstract This paper considers the limit of the ratio of two incomplete beta functions $I_x(p+s,q+r)$ to $I_x(p,q)$ as $p+q \to \infty$. The results show that the limits depend on r,s,x and the limit of $p/(p+q)$.

1. Introduction

The form of the beta probability density function of X with parameters p, q is

$$f_X(x)\frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}, \quad 0 \le x \le 1\, f_X(x) = \frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}, \quad 0 \le x \le 1. \tag{1}$$

with p, q>0, where $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$.

The beta density function can assume a variety of both symmetrical and asymmetrical shapes. If $p=q$, the distribution is symmetrical about x=1/2. The probability integral of the distribution (1) up to x is called the incomplete beta function and denoted by $I_x(p,q)$, so that

$$I_x(p,q) = \frac{1}{B(p,q)}\int_0^x x^{p-1}(1-x)^{q-1}dx \tag{2}$$

If this ratio is kept constant, but p and q both increased, the variance decreases, and the standardized distribution tends to the unit normal distribution. Many approximations to $I_x(p,q)$ are given in Johnson and Kotz(1970, pp. 41-51).

In this paper we obtain the limit of the ratio;

$$\rho_{r,s}(\mu, x) = \lim_{n\to\infty}\frac{I_x(p+r,q+s)}{I_x(p,q)}, \quad 0 \le x \le 1 \tag{3}$$

where $p/(p+q) \to \mu$ as $n = p+q \to \infty$

2. Limit of the ratio

Here we investigate the limiting behavior of the ratio of two incomplete beta functions given in (3). Henceforth, we replace p and q with p+1 and q+1, respectively, for

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expressional convenience.

Lemma 1. Let $p+q \to \infty$ such that $p/(p+q) \to \mu$ as $n \to \infty$. Then, for any π_1 and π_2 such that $0 < \pi_1 < \pi_2 < \mu$,

$$\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} \to x^r(1-x)^s \tag{4}$$

uniformly for x on $[\pi_1, \pi_2]$.

Proof. Let $\varepsilon > 0$ be arbitrary. Choose $\Delta x > 0$ so that $\Delta x < \pi_1/2$ and $|x_1^r(1-x)^s - x_2^r(1-x)^s| < \varepsilon/3$ if $|x_1 - x_2| \le \Delta x$ and $|x_1, x_2| \in [\pi_1/2, \pi_2]$. Choose $k > 0$ so that $k < \mu - \pi_2$, and choose $N_1 > 0$ so that $|p/(p+q) - \mu| < k$ if $n \ge N_1$. Note that for fixed $n, u^p(1-u)^q$ is increasing on $[0, p/(p+s)]$. For $n \ge N_1$ and $x \in [\pi_1, \pi_2]$,

$$\left| \frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} - x^r(1-x)^s \right|$$

$$\le \frac{\int_0^{x-\Delta x} u^p(1-u)^q |u^r(1-u)^s - x^r(1-x)^s| du}{\int_0^x u^p(1-u)^q du}$$

$$+ \frac{\int_{x-\Delta x}^x u^p(1-u)^q |u^r(1-u)^s - x^r(1-x)^s| du}{\int_0^x u^p(1-u)^q du}$$

$$\le \max_{x_1, x_2 \in [0, x-\Delta x]} |x_1^r(1-x_1)^s - x_2^r(1-x_2)^s| \frac{\int_0^{x-\Delta x} u^p(1-u)^q du}{\int_0^x u^p(1-u)^q du}$$

$$+ \max_{x_1 \in [x-\Delta x, x]} |x_1^r(1-x_1)^s - x^r(1-x)^s| \frac{\int_{x-\Delta x}^x u^p(1-u)^q du}{\int_0^x u^p(1-u)^q du}$$

$$< M_1 \frac{\int_0^{x-\Delta x} u^p(1-u)^q du}{\int_0^x u^p(1-u)^q du} + \frac{\epsilon}{3}$$

$$< M_1 \frac{\int_0^{x-x'} u^p(1-u)^q du + \int_{x-x'}^{x-\Delta x} u^p(1-u)^q du}{\int_{x-\Delta x}^x u^p(1-u)^q du} + \frac{\epsilon}{3}$$

$$< M_1 \frac{(x-x')(x-x')^p[1-(x-x')]^q}{\Delta x(x-\Delta x)^p[1-(x-\Delta x)]^q} + M_1 \frac{\frac{\epsilon}{3M_1}\Delta x(x-\Delta x)^p[1-(x-\Delta x)]^q}{\Delta x(x-\Delta x)^p[1-(x-\Delta x)]^q} + \frac{\epsilon}{3} \tag{5}$$

where $M_1 = \max_{x_1,x_2 \in [0,x-\Delta x]} |x_1^r(1-x_1)^s - x_2^r(1-x_2)^s|$ and $x' = \Delta x + \frac{\epsilon}{3M_1}\Delta x$. Since $u^p(1-u)^q$ is increasing on $[0, p/(p+q)]$ and $x < p/(q+p)$, we have

$$\text{right hand side(rhs) of (5)} = M_2\left(\frac{x-x'}{x-\Delta x}\right)^p\left(\frac{1-(x-x')}{1-(x-\Delta x)}\right)^q + \frac{2\epsilon}{3} \tag{6}$$

where $M_2 = M_1(x-x')/\Delta x$. Since $(x-x')/(x-\Delta x) \le (\pi_2 - x')/(\pi_2 - \Delta x)$ and $(1-x+x')/(1-x+\Delta x) \le (1-\pi_2+x')/(1-\pi_2+\Delta x)$, we have

$$\begin{aligned}
\text{rhs of (6)} &\le M_2\left(\frac{\pi_2-x'}{\pi_2-\Delta x}\right)^p\left(\frac{1-(\pi_2-x')}{1-(\pi_2-\Delta x)}\right)^q + \frac{2\epsilon}{3} \\
&= M_2\left[\left(\frac{\pi_2-x'}{\pi_2-\Delta x}\right)^{\frac{p}{p+q}}\left(\frac{1-(\pi_2-x')}{1-(\pi_2-\Delta x)}\right)^{\frac{q}{p+q}}\right]^{p+q} + \frac{2\epsilon}{3} \\
&< M_2\left[\left(\frac{\pi_2-x'}{\pi_2-\Delta x}\right)^{\mu-k}\left(\frac{1-\pi_2+x'}{1-\pi_2+\Delta x}\right)^{1-\mu+k}\right]^{p+q} + \frac{2\epsilon}{3}
\end{aligned} \tag{7}$$

where in the final inequality, we use $p/(p+q) > \mu - k$, and $(\pi_2 - x')/(\pi_2 - \Delta x) < 1$. Defining $f_1(t) = (t/(\pi_2-\Delta x))^{\mu-k} \cdot ((1-t)/(1-\pi_2+\Delta x))^{1-\mu+k}$, , we have $f_1(t)$ is increasing on $[0, \mu-k]$ and $f_1(\pi_2 - x') < f_2(\pi_2 - \Delta x) = 1$. Thus, there exists $N \ge N_1$ such that the first term of rhs of (7) is less than $\varepsilon/3$ if $n \ge N$. Hence, for all $n \ge N$ and $x \in [\pi_1, \pi_2]$,

$$\left| \frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} - x^r(1-x)^s \right| < \varepsilon \tag{8}$$

Lemma 2. For any π_3 such that $\mu < \pi_3 < 1$,

$$\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} \to \mu^r(1-\mu)^s \tag{9}$$

uniformly for x on $[\pi_3, 1]$ as $p+q \to \infty$.

Proof. Choose $N > 0$ so that $|p/(p+q) - \mu| < (\pi_3 - \mu)/3$ if $n \ge N$. For $n \ge N$ and $x \in [\pi_3, 1]$,

$$\left|\frac{\int_0^x u^p(1-u)^q du}{\int_0^1 u^p(1-u)^q du}-1\right|=\frac{\int_x^1 u^p(1-u)^q du}{\int_0^1 u^p(1-u)^q du}<\frac{\int_{\pi_3}^1 u^p(1-u)^q du}{\int_{(2\mu+\pi_3)/3}^{(\mu+2\pi_3)/3} u^p(1-u)^q du} \tag{10}$$

We note that $u^p(1-u)^q$ is decreasing on $[p/(p+q),1]$. Since $p/(p+q)<(2\mu+\pi_3)/3$ if $n\geq N$ and $\pi_3/((\mu+2\phi_3)/3)>1$, it follows that

$$\text{rhs of inequality (10)} < \frac{(1-\pi_3)\pi_3^p(1-\pi_3)^q}{\frac{\pi_3-\mu}{3}(\frac{\mu+2\pi_3}{3})^p(1-\frac{\mu+2\pi_3}{3})^q}$$

$$=3\frac{1-\pi_3}{\pi_3-\mu}\left[\left(\frac{\pi_3}{\frac{\mu+2\pi_3}{3}}\right)^{\frac{p}{q+p}}\left(\frac{1-\pi_3}{1-\frac{\mu+2\pi_3}{3}}\right)^{\frac{q}{q+p}}\right]^{q+p}$$

$$<3\frac{1-\pi_3}{\pi_3-\mu}\left[\left(\frac{\pi_3}{\frac{\mu+2\pi_3}{3}}\right)^{\mu+\xi}\left(\frac{1-\pi_3}{1-\frac{\mu+2\pi_3}{3}}\right)^{1-\mu-\xi}\right]^{q+p} \tag{11}$$

where $\xi=(\pi_3-\mu)/3$. Defining $f_2(t)=(t/(\frac{\mu+2\pi_3}{3}))^{\mu+\xi}((1-t)/(1-\frac{\mu+2\pi_3}{3}))^{1-\mu-\xi}$, we have $f_2(t)$ is decreasing on $[\mu+\xi,1]$ and $f_2(\pi_3)<f_2(\frac{\mu+2\pi_3}{3})=1$, thus

$$\frac{\int_0^x u^p(1-u)^q du}{\int_0^1 u^p(1-u)^q du}\to 1$$

uniformly on $[\pi_3,1]$ as $p+q\to\infty$. For the same reason,

$$\frac{\int_0^x u^{p+r}(1-u)^{q+s} du}{\int_0^1 u^{p+r}(1-u)^{q+s} du}\to 1 \tag{12}$$

uniformly on $[\pi_3,1]$ Hence

$$\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du}=\frac{\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^1 u^{p+r}(1-u)^{q+s}du}}{\frac{\int_0^x u^p(1-u)^q du}{\int_0^1 u^p(1-u)^q du}}\cdot\frac{\int_0^1 u^{p+r}(1-u)^{q+s}du}{\int_0^1 u^p(1-u)^q du}\to\mu^r(1-\mu)^s \tag{13}$$

uniformly on $[\pi_3,1]$ as $p+q\to\infty$. This completes the proof.

Theorem 1. Let $p+q\to\infty$ such that $p/(p+q)\to\mu$, as $n\to\infty$. Then

$$\frac{I(x;p+r,q+s)}{I(x;p,q)}\to\frac{x^r(1-x)^s}{\mu^r(1-\mu)^s}I[x<\mu]+I[x\geq\mu] \tag{14}$$

uniformly on any compact subset of (0,1] as $p+q\to\infty$, where I[·] denotes the usual indicator function.

Proof. It suffices to show that for π_1 such that $0 < \pi_1 < \mu$,

$$\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} \to f_3(x)$$

uniformly on $[\pi_1, 1]$ where $f_3(x) = x^r(1-x)^s I[0 < x \le \mu] + \mu^r(1-\mu)^s I[\mu < x \le 1]$. For given $\varepsilon > 0$ choose π_2 and π_3 so that $\pi_1 < \pi_2 < \mu < \pi_3 < 1$ and $|x^r(1-x)^s / \mu^r(1-\mu)^s - 1| \le \varepsilon$. Choose N>0 such that Lemmas 1 and 2 hold if $n \ge N$. For $n \ge N$, $x \in [\pi_2, \pi_3]$,

$$\begin{aligned}
&\int_0^x u^{p+r}(1-u)^{q+s}du \\
&\quad < (\pi_2^r(1-\pi_2)^s + \varepsilon)\int_0^{\pi_2} u^p(1-u)^q du \\
&\qquad + (1+\varepsilon)\mu^r(1-\mu)^s \int_{\pi_2}^x u^p(1-u)^q du \\
&\quad < [(1+\varepsilon)\mu^r(1-\mu)^s + \varepsilon]\int_0^{\pi_2} u^p(1-u)^q du \\
&\qquad + (1+\varepsilon)\mu^r(1-\mu)^s \int_{\pi_2}^x u^p(1-u)^q du \\
&\quad < [1 + (1 + \tfrac{1}{\mu^r(1-\mu)^s})\varepsilon]\mu^r(1-\mu)^s \int_0^x u^p(1-u)^q du
\end{aligned}$$

Hence

$$\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} - \mu^r(1-\mu)^s < [1 + \mu^r(1-\mu)^s]\varepsilon \tag{15}$$

Similarly, we can show that

$$\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} - \mu^r(1-\mu)^s > -[1 + \mu^r(1-\mu)^s]\varepsilon \tag{16}$$

Therefore, for n >N,

$$
\begin{aligned}
&\left|\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} - f_3(x)\right| \\
&\leq \left|\frac{\int_0^x u^{p+r}(1-u)^{q+s}du}{\int_0^x u^p(1-u)^q du} - \mu^r(1-\mu)^s\right| \\
&\quad + |\mu^r(1-\mu)^s - f_3(x)| \\
&< [1+\mu^r(1-\mu)^s]\varepsilon + \mu^r(1-\mu)^s\varepsilon \\
&= [1 = 2\mu^r(1-\mu)^s]\varepsilon
\end{aligned} \tag{17}
$$

Combining Lemmas 1,2, and (17), we have (14).

Reference

Johnson, No. L. and Kotz, S., (1970), *Distributions in statistics, continuous univariate distributions-2*, Houghton Miflin company.