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Tail Probability Approximations for the Ratio of the Independent Random Variables

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Abstract In this paper, we study the saddlepoint approximations for the ratio of independent random variables. In Section 2, we derive the saddlepoint approximation to the density. And in Section 3, we derive two approximation formulae for the tail probability, one by following Daniels' (1987) method and the other by following Lugannani and Rice's (1980). In Section 4, we represent some numerical examples which show that the errors are small even for small sample size.

Keywords : Saddlepoint approximation, Probability density function, Tail probability.

1. Introduction

It is often required to approximate the distribution of some statistics whose distribution can not be exactly obtained.

When the first few moments are known, a common procedure is to fit a law of Edgeworth type having the same moments as far as they are given (Edgeworth (1905), Wallace (1958)). This method is often satisfactory in practice, but can assume negative values in the far tail region of distribution.

Daniels (1954) introduced a new type of idea into statistics by applying saddlepoint techniques to derive a very accurate approximation to the distribution of $\bar{X}$. He showed that the error incurred by using the saddlepoint approximation method is $O(n^{-1})$ as against the more usual $O(n^{-1/2})$ associated with the normal approximation. Moreover, he showed that the relative error of the approximation is uniformly $O(n^{-1})$ over the whole range of the random variable in an important class of cases. For reviews of saddlepoint approximations, see Reid (1988) and Field and Ronchetti (1990).

In this paper, we study the saddlepoint approximations for the ratio of independent random variables. In Section 2, we derive the saddlepoint approximation to the density. And in Section 3, we derive two approximation formulae for the tail probability, one by following Daniels' (1987) method and the other by following Lugannani and Rice's

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(1980). In Section 4, we represent some numerical examples which show that the errors are small even for small sample size.

Let $\{U_n, n \geq 1\}, \{S_n > 0, n \geq 1\}$ be independent sequences of random variables with absolutely continuous distribution functions F_{1n}, F_{2n} respectively. Denote $\phi_{1n}(t) = E\{\exp(tU_n)\}$ and $\phi_{2n}(t) = E\{\exp(tS_n)\}$ be the moment generating functions of U_n and S_n, respectively. And let $\psi_{1n}(t) = (1/n)\log\phi_{1n}(t)$, and $\psi_{2n}(t) = (1/n)\log\phi_{2n}(t)$ be their cumulant generating function. Assume that $\phi_{1n}(t)$ and $\psi_{1n}(t)$ exist for real t in some interval (t_1, t_2) containing 0 and that $\phi_{2n}(t)$ and $\psi_{2n}(t)$ exist for real t in some interval (t_3, t_4) containing 0. Let $R_n = U_n / S_n$.

2. Density approximations

The fact that the integrand in Fourier inversion the density of R_n is of the form $\exp[n\{\Psi(z)\}]$ is the starting point to derive the saddlepoint approximations for R_n.

Let H_n be the distribution function of R_n. Then

$$H_n(r) = P_r(R_n \leq r) = \int_0^{+\infty} F_{1n}(ry)dF_{2n}(y). \tag{1}$$

And the p.d.f. h_n of R_n is given by

$$h_n(r) = \int_0^{+\infty} yf_{1n}(ry)dF_{2n}(y). \tag{2}$$

where f_{1n} is the p.d.f. of U_n.

The characteristic function of R_n is given by

$$\begin{aligned} \hat{h}_n(t) &= \int_{-\infty}^{+\infty} e^{itr} \int_0^{+\infty} yf_{1n}(ry)dF_{2n}(y)dr \\ &= \int_{-\infty}^{+\infty} \phi_{1n}\left(\frac{it}{y}\right) dF_{2n}(y). \end{aligned} \tag{3}$$

Using the Fourier inversion formula, the p.d.f. h_n is given by

$$\begin{aligned} h_n(r) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itr} \hat{h}_n(t)dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \int_0^{+\infty} \phi_{1n}\left(\frac{it}{y}\right) dF_{2n}(y) \right\} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi_{1n}(is) \left\{ \int_0^{+\infty} e^{-isyr} \cdot ydF_{2n}(y) \right\} ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi_{1n}(it)\phi'_{2n}(-irt)dt \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi_{1n}(z)\phi'_{2n}(-rz)dz$$

$$= \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[n\{\psi_{1n}(z)+\psi_{2n}(-rz)\}]\times\psi'_{2n}(-rz)dz \tag{4}$$

where c is any real number in the interval where the moment generating function exists. When n is large, an approximation is found by passing the path of integration through a saddlepoint τ of the exponential part of integrand given by $\psi'_{1n}(\tau) - r\psi'_{2n}(-r\tau) = 0$. We choose c to be τ.

On the contour near τ, we have

$$\begin{aligned} n\{\psi_{1n}(z)+\psi_{2n}(-rz)\} &= n\Psi_n(z)(\text{say}) \\ &= n\left\{\Psi_n(\tau) + \frac{\Psi''(\tau)}{2}(z-\tau)^2 + \frac{\Psi^{(3)}(\tau)}{6}(z-\tau)^3 + \cdots\right\} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \psi'_{2n}(-rz) &= \psi'_{2n}(-r\tau) - r\psi^{(2)}_{2n}(-r\tau)(z-\tau) \\ &\quad + \frac{r^2}{2}\psi^{(3)}_{2n}(-r\tau)(z-\tau)^2 - \frac{r^3}{6}\psi^{(4)}_{2n}(-r\tau)(z-\tau)^3 + \cdots. \end{aligned} \tag{6}$$

Let $\sqrt{n\Psi''_n(\tau)}(z-\tau) = iy$ and expanding the integrand in (4) near τ, we have

$$\begin{aligned} h_n(r) &= \left(\frac{n}{\hat{\Psi}''_n}\right)^{\frac{1}{2}} \frac{\exp[n\{\psi_{1n}(\tau)+\psi_{2n}(-r\tau)\}]}{2\pi}\psi'_{2n}(-r\tau) \\ &\quad \times \int_{-\infty}^{+\infty}\exp\left(\frac{y^2}{2}\right)\left(1 - \frac{\lambda_3}{6\sqrt{n}}iy^3 + \frac{\lambda_4}{24n}y^4 - \frac{\lambda_3^2}{72n}y^6 + \cdots\right) \\ &\quad \times \left\{1 - \frac{r\psi^{(2)}_{2n}(-r\tau)}{\psi'_{2n}(-r\tau)}\frac{iy}{\sqrt{n}(\hat{\Psi}''_n)^{\frac{1}{2}}} - \frac{r^2\psi^{(3)}_{2n}(-r\tau)}{2n\psi'_{2n}(-r\tau)\hat{\Psi}''_n}y^2\right. \\ &\quad \left. + \frac{r^3\psi^{(4)}_{2n}(-r\tau)}{6n\sqrt{n}\psi'_{2n}(-r\tau)(\hat{\Psi}''_n)^{\frac{3}{2}}}y^3 + \cdots dy\right\} \\ &= \tilde{h}_n(r)\left[1 + \frac{3}{n}\left\{\frac{\lambda_4}{24} - \frac{r\psi^{(2)}_{2n}(-r\tau)\lambda_3}{6\psi'_{2n}(-r\tau)(\hat{\Psi}''_n)^{\frac{1}{2}}}\right\} - \frac{15\lambda_3^2}{72n}\right. \end{aligned}$$

$$-\frac{r^2\psi_{2n}^{(3)}(-r\tau)}{2n\psi'_{2n}(-r\tau)\hat{\Psi}''_n}+\cdots\Bigg]. \qquad (7)$$

where

$$\tilde{h}_n(r)=\frac{\sqrt{n}\psi'_{2n}(-r\tau)\exp[n\{\psi_{1n}(\tau)+\psi_{2n}(-r\tau)\}]}{\sqrt{2\pi\{\psi''_{1n}(\tau)+r^2\psi_{2n}(-r\tau)\}}} \qquad (8)$$

and

$$\hat{\Psi}_n^{(j)}=\Psi_n^{(j)}(\tau), \quad \lambda_j=\frac{\hat{\Psi}_n^{(j)}}{(\hat{\Psi}''_n)^{j/2}}.$$

We call $\tilde{h}_n(r)$ the saddlepoint approximation to $h_n(r)$.

3. Tail probability approximations

In this section, We are interested in the tail probability of R_n, i.e., $P_r(R_n \geq r)=\overline{H}_n(r)$ and derive two approximation formulae for it. The tail probability can be approximated by integrating the saddlepoint approximation $\tilde{h}_n(r)$ numerically, i.e.,

$$\tilde{\overline{H}}_n(r)=\int_r^{\infty}\tilde{h}_n(y)dy. \qquad (9)$$

To obtain the explicit approximate formula for the tail probability, we must consider the next inversion formula.

$$\overline{H}_{1n}(r)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[n\{\psi_{1n}(z)+\psi_{2n}(-rz)\}]\frac{dz}{z} \quad (c>0). \qquad (10)$$

The above relation (10) for the tail probability of R_n is obtained as follows. Since the p.d.f of R_n is

$$h_n(r)=\frac{n}{2\pi}\int_{-\infty}^{+\infty}\exp[n\{\psi_{1n}(c+it)+\psi_{2n}(-r(c+it))\}]\times\psi'_{2n}\{-r(c+it)\}dt \qquad (11)$$

$$\begin{aligned}\overline{H}_n(r)&=\frac{n}{2\pi}\int_r^{+\infty}\int_{-\infty}^{+\infty}\exp[n\{\psi_{1n}(c+it)+\psi_{2n}(-r(c+it))\}]\times\psi'_{2n}\{-r(c+it)\}dtdr\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[n\{\psi_{1n}(c+it)\}[-\frac{1}{c+it}\exp\{n\{\psi_{2n}(-r(c+it))\}|_r^{\infty}]dt\end{aligned} \qquad (12)$$

Since $\exp\{n\{\psi_{2n}(-r(c+it))\}=\phi_{2n}\{-r(c+it)\}=\int_0^{\infty}\exp\{-r(c+it)x\}\times dF_{2n}(x)\}$ converges to zero as $r\to\infty$ and $c>0$, the relation (10) is obtained. As in Daniels (1987), expanding only $\Psi_n(z)=\psi_{1n}(z)+\psi_{2n}(-rz)$ and integrating, we have

$$\overline{H}_n(r) = \exp(n\hat{\Psi}_n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\left\{\frac{1}{2}\hat{\Psi}_n''(z-\tau)^2\right\}$$
$$\times\left\{1+\frac{1}{6}\hat{\Psi}_n^{(3)}(z-\tau)^3+\frac{1}{24}n\hat{\Psi}_n^{(4)}(z-\tau)^4\right.$$
$$\left.+\frac{1}{72}n^2(\hat{\Psi}_n^{(3)})^2(z-\tau)^6+\cdots\right\}\frac{dz}{z}$$
$$= \exp\left(n\hat{\Psi}_n+\frac{1}{2}\hat{u}^2\right)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\left(\frac{1}{2}u^2-\hat{u}u\right)$$
$$\times\left[1+\frac{\lambda_3}{6\sqrt{n}}(u-\hat{u})^3+\frac{1}{n}\left\{\frac{\lambda_4}{24}(u-\hat{u})^4+\frac{\lambda_3^2}{72}(u-\hat{u})^6\right\}+\cdots\right]\frac{du}{u} \tag{13}$$

where $u = z(n\hat{\Psi}_n'')^{1/2}$ and $\lambda_j = \hat{\Psi}_n^{(j)} / (\hat{\Psi}_n'')^{j/2}$, $j \geq 3$. The above tail probability $\overline{H}_n(r)$ can be found from the fact that $I_r = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(\frac{1}{2}u^2 - \hat{u}u)(u-\hat{u})^r du / u$ satisfies the recurrence relations :

$$I_{2m} = (-\hat{u})I_{2m-1}, \quad I_{2m+1} = (-1)^m a_m\phi(\hat{u}) - \hat{u}I_{2m}$$
$$\text{with } I_0 = 1-\Phi(\hat{u}), \quad a_0 = 1 \text{ and } a_m = 1\times\cdots\times(2m-1) \tag{14}$$

Repeated applications of (14) lead to the explicit formula

$$I_r = (-\hat{u})^r\{1-\Phi(\hat{u})\} + (-1)^{r-1}\phi(\hat{u})\sum_{m=0}^{[\frac{1}{2}(r-1)]}(-1)^m a_m\hat{u}^{r-2m-1} \tag{15}$$

In (13), $I_3 = -\hat{u}^3\{1-\Phi(\hat{u})\} + (\hat{u}^2-1)\phi(\hat{u})$ gives the following formula.

$$\overline{H}_n(r) = \exp\left(n\hat{\Psi}_n+\frac{1}{2}\hat{u}^2\right)\times\left[\{1-\Phi(\hat{u})\}\left(1-\frac{\lambda_3\hat{u}^3}{6\sqrt{n}}\right)\right.$$
$$\left.+\phi(\hat{u})\frac{\lambda_3}{6\sqrt{n}}\times(\hat{u}^2-1)\right]\times\{1+O(n^{-1})\} \tag{16}$$
$$= \tilde{\overline{H}}_n^1(r)\{1+O(n^{-1})\} \text{ (say).}$$

And substitution of I_4 and I_6 in (13) gives the following formula;

$$\overline{H}_n(r) = \exp\left(n\hat{\Psi}_n+\frac{1}{2}\hat{u}^2\right)\left[\{1-\Phi(\hat{u})\}\left\{1-\frac{\lambda_3\hat{u}^3}{6n^{\frac{1}{2}}}+\frac{1}{n}\left(\frac{\lambda_4\hat{u}^4}{24}+\frac{\lambda_3^2\hat{u}^6}{72}\right)\right\}\right.$$
$$\left.+\phi(\hat{u})\left\{\frac{\lambda_3}{6n^{\frac{1}{2}}}(\hat{u}^2-1)-\frac{1}{n}\left(\frac{\lambda_4}{24}(\hat{u}^3-\hat{u})+\frac{\lambda_3^2}{72}(\hat{u}^5-\hat{u}^3+3\hat{u})\right)\right\}\right]$$
$$\times\{1+O(n^{-\frac{3}{2}})\}$$

$$= \widetilde{\overline{H}}_n^2(r)\{1 + O(n^{-\frac{3}{2}})\} \text{ (say)}. \tag{17}$$

When $\tau \le 0$, we can also use the following inversion formula

$$\overline{H}_n(r) = h(-\tau) + \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \exp\{n\psi_n(z)\}\frac{dz}{z}. \tag{18}$$

where $h(x) = 0, 1/2, 1,$ when $x < 0, = 0, > 0$, respectively. See Daniels(1987) p 43. So, (16) can be replaced by

$$\overline{H}_n(r) = h(-\hat{u}) + \exp(n\hat{\Psi}_n + \frac{1}{2}\hat{u}^2) \times [\{h(-\hat{u}) - \Phi(\hat{u})\}\{1 - \frac{\lambda_3\hat{u}^3}{6\sqrt{n}}\}$$
$$+ \phi(\hat{u})\frac{\lambda_3}{6\sqrt{n}} \times (\hat{u}^2 - 1)] \times \{1 + O(n^{-1})\}$$
$$= \widetilde{\overline{H}}_n^1(r) \times \{1 + O(n^{-1})\} \text{ (say)}. \tag{19}$$

And (15) can be replaced by

$$\overline{H}_n(r) = h(-\hat{u}) + \exp\{n\hat{\Psi}_n + \frac{1}{2}\hat{u}^2\}\Bigg[\{h(-\hat{u}) - \Phi(\hat{u})\}\{1 - \frac{\lambda_3\hat{u}^3}{6n^{\frac{1}{2}}}$$
$$+ \frac{1}{n}\left(\frac{\lambda_4\hat{u}^4}{24} + \frac{\lambda_3^2\hat{u}^6}{72}\right)\Bigg\} + \phi(\hat{u})\Bigg[\frac{\lambda_3}{6n^{\frac{1}{2}}}(\hat{u}^2 - 1) - \frac{1}{n}\Bigg\{\frac{\lambda_4}{24}(\hat{u}^3 - \hat{u})$$
$$+ \frac{\lambda_3^2}{72}(\hat{u}^5 - \hat{u}^3 + 3\hat{u})\Bigg\}\Bigg]\Bigg] \times \{1 + O(n^{-\frac{3}{2}})\}$$
$$= \widetilde{\overline{H}}_n^2(r)\{1 + O(n^{-\frac{3}{2}})\} \text{ (say)}. \tag{20}$$

We can also obtain another expression valid for any location of τ, that is an expression which is uniform in τ. Consider the inversion formula for tail probability once more.

$$\overline{H}_n(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{n\psi_n(z)\}\frac{dz}{z}, \quad (c > 0). \tag{21}$$

which has a pole at the origin.

As in Lugannani and Rice (1980), we can use the Bleistein's(1966) idea which has been developed for integrals with the saddlepoint near algebraic singularity, it is a simple pole in our case. Following the similar procedures in M.S Srivastiva and Wai Kwok Yao(1988), we can obtain another tail probability approximation formula .

The basic idea is to find a transformation which takes account of the proximity of the saddlepoint and simple pole for small values of z. Such transformation can be accomplished by putting

$$\psi_{1n}(z) + \psi_{2n}\{-r(\tau)z\} = \frac{1}{2}w^2 - \hat{w}w \tag{22}$$

where $\hat{w} = \text{sgn}(\tau)(-2[\psi_{1n}(\tau) + \psi_{2n}\{-r(\tau)\tau\}])^{\frac{1}{2}}$ and $r(\tau)$ is defined as $\psi'_{1n}(\tau) - r(\tau)\psi'_{2n}\{-r(\tau)\tau\} = 0$, which implies that $\hat{w} = 0$ when $\tau = 0$, which, in turn, implies that $\psi_{1n}(z) + \psi_{2n}\{-r(0)z\} = \frac{1}{2}w^2$. Therefore the origin remains fixed, i.e., at $z = 0, w(0) = 0.$

By differentiating (22) w.r.t. w, we obtain

$$\frac{dz}{dw} = \frac{w - \hat{w}}{\psi'_{1n}(z) - r(\tau)\psi'_{2n}\{-r(\tau)z\}} \tag{23}$$

If $\tau = 0$, i.e., $r = E(U_n) / E(S_n)$, then we have

$$\frac{dw}{dz} = \left(\frac{dz}{dw}\right)^{-1} = \frac{\psi'_{1n}(z) - r(\tau)\psi'_{2n}\{-r(\tau)z\}}{w} \tag{24}$$

Using L'hospital's rule, we have

$$\frac{dw}{dz}|_{z=0} = \{\psi''_{1n}(0) + r^2\psi''_{2n}(0)\}^{\frac{1}{2}} \tag{25}$$

However $\tau \neq 0$, i.e., $r \neq E(U_n) / E(S_n)$

$$\frac{dw}{dz}|_{z=0} = \frac{r(\hat{z})\psi'_{2n}(0) - \psi'_{1n}(0)}{\hat{w}} \tag{26}$$

Thus we obtain

$$c = w'(0) = \begin{cases} \{r(\tau)\psi'_{2n}(0) - \psi'_{1n}(0)\} / \hat{w}, \text{if } r = E(U_n) / E(S_n) \\ \{\psi''_{1n}(0) + r^2\psi''_{2n}(0)\}^{\frac{1}{2}}, \quad \text{if } r \neq E(U_n) / E(S_n), \end{cases} \tag{27}$$

and that

$$\begin{aligned} w = w(z) &= w(0) + w'(0)z + \frac{w^{(2)}(\theta z)}{2}z^2, \quad \text{for } 0 < \theta < 1 \\ &= 0 + cz + o(z), \quad \text{as } z \to 0 \end{aligned} \tag{28}$$

From which, we can see that $w \sim cz$ when z is small, which implies $z / w \sim dz / dw \sim c^{-1}$ or $z^{-1}(dz / dw) \sim w^{-1}$ when z is small. Since dz / dw is analytic in neighborhood of w=0, so is $z^{-1}(dz / dw) - w^{-1}$

Using the transformation given in (22), the formula (21) transforms to

$$\overline{H}_n(r) = \int_{\hat{w}-i\infty}^{\hat{w}+i\infty} \exp\{n(\frac{1}{2}w^2 - \hat{w}w)\}\left(\frac{1}{z}\frac{dz}{dw}\right)dw$$

$$= \frac{1}{2\pi i}\int \exp\{n(\frac{w^2}{2} - \hat{w}w)\}\frac{dw}{w}$$

$$+ \frac{\exp(-\frac{1}{2}n\hat{w}^2)}{2\pi i}\int_{\hat{w}-i\infty}^{\hat{w}+i\infty} \exp\{\frac{1}{2}(w-\hat{w})^2\}\left(\frac{1}{z}\frac{dz}{dw} - \frac{1}{w}\right)dw$$

$$= I_1 + I_2 \text{ (say)} \tag{29}$$

To find the value of I_1, we need only to see that it is indeed the inversion formula for the tail probability $P_r(\overline{X} \geq \hat{w}) = \overline{\Phi}(-\sqrt{n}\hat{w})$, where $\overline{X}$ is the sample mean from i.i.d. N(0 , 1)

And following the arguments as in M.S. Srivastava and W.K. Yao(1988) we obtain the relation that

$$I_2 \sim \phi(\sqrt{n}\hat{w})\left(\frac{a_{0n}}{\sqrt{n}} - \frac{a_{2n}}{n\sqrt{n}} + \cdots\right) \tag{30}$$

where

$$a_{0n} = \frac{1}{\hat{\mu}} - \frac{1}{\hat{w}},\ a_{2n} = \left(\frac{5\lambda_3^2}{24} - \frac{\lambda_4}{8}\right)\frac{1}{\hat{\mu}} + \frac{\lambda_3}{2\hat{\mu}^2} + \frac{1}{\hat{\mu}^3} - \frac{1}{\hat{w}^3},\ \hat{\mu} = \tau(\hat{\Psi}_n'')^{\frac{1}{2}}$$

and $a_{2k,n} = O(1)$ for all $k \geq 0$. Thus, by substituting the above results in (27) we have the following approximation formula for the ratio R_n;

$$\overline{H}_n(r) \sim \overline{\Phi}(\sqrt{n}\hat{w}) + \phi(\sqrt{n}\hat{w})\left(\frac{a_{0n}}{\sqrt{n}} - \frac{a_{2n}}{n\sqrt{n}} + \cdots\right)$$

$$= \overline{\Phi}(\sqrt{n}\hat{w}) + \phi(\sqrt{n}\hat{w})\frac{a_{0n}}{\sqrt{n}} + O(n^{-\frac{3}{2}})$$

$$= \overline{H}_n^{LR}(r) + O(n^{-\frac{3}{2}}) \text{ (say)} \tag{31}$$

where a_{m}'s are the same as above.

In (31), $a'_{sk,n}s(k \geq 0)$ are uniformly bounded as $\tau, \hat{\mu}$ and $\hat{w}$ cross the origin (see Srivastiva and Yao (1988)).

4. Numerical example

In this section, we present an example to show that the errors of our approximations formulae are small.

Example 4.1. Assume that $\{U_n\}$ and $\{S_n\}$ have $\chi^2(n)$ distribution and are independent. Then $R_n = U_n / S_n$ follows F-distribution with (n, n) degrees of freedom. By definition, we have

$$\phi_{in}(z) = (1-2z)^{-\frac{n}{2}},\ \psi_{in}(z) = -\frac{1}{2}\log(1-2z),$$
$$\psi'_{in}(z) = (1-2z)^{-1} \text{ and } \psi_{in}^{(2)} = 2(1-2z)^{-2} (i = 1,2).$$

So the saddlepoint equation becomes $\psi'_{in}(\tau) - r\psi'_{2n}(-r\tau) = 0$ and $\tau = (r-1)/(4r)$. Therefore the saddlepoint approximation to the p.d.f. of R_n is given by

$$\tilde{h}_n(r) = \frac{\sqrt{n}\psi'_{2n}(-r\tau)\exp[n\{\psi_{1n}(\tau) + \psi_{2n}(-r\tau)\}]}{[2\pi\{\psi''_{1n}(\tau) + r^2\psi''_{2n}(-r\tau)\}]^{\frac{1}{2}}}$$
$$= \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \frac{2^{n-1} r^{\frac{1}{2}-1}}{(1+r)^n} \tag{32}$$

The exact p.d.f. of R_n is well known as

$$h_n(r) = \frac{\Gamma(n) r^{\frac{n}{2}-1}}{\Gamma^2(n/2)(1+r)^n} \tag{33}$$

Note that the ratio $B_n = \tilde{h}_n(r)/h_n(r) = \sqrt{2\pi/n}/\{\Gamma^2(n/2)2^{n-1}\}$ does not depend on r and is nearly 1 as n increases. We can see that the error is uniformly small in this case.
We can obtain the exact tail probability by integrating $h_n(r)$ from r to ∞, i.e.,

$$\overline{H}_n(r) = \int_r^\infty \frac{\Gamma(n) x^{\frac{n}{2}-1}}{\Gamma^2(n/2)(1+x)^n} dx \tag{34}$$

From (16) and (17) we obtain two approximation formulae for the tail probability of R_n, i.e., Since

$$\hat{\Psi}''_n = \frac{16r^2}{(r+1)^2},\ \hat{\Psi}_n^{(3)} = 0,\ \hat{\Psi}_n^{(4)} = \frac{2^5 \cdot 48 \cdot r^4}{(r+1)^4},\ \lambda_3 = 0,\ \lambda_4 = 6$$

$$\tilde{\overline{H}}{}^1_{1n}(r) = h(-\hat{z}) + \exp\left\{-\frac{n}{2}\log\frac{(r+1)^2}{4r} + \frac{n}{2}\left(\frac{r-1}{r+1}\right)^2\right\}$$
$$\times \left\{h(-\hat{z}) - \Phi\left(\sqrt{n}\frac{r-1}{r+1}\right)\right\} \tag{35}$$

and

$$\widetilde{\widetilde{H}}_{2n}^{2}(r) = h(-\hat{z}) + \exp\left\{-\frac{n}{2}\log\left\{\frac{(r+1)^2}{4r}\right\} + \frac{n}{2}\left(\frac{r-1}{r+1}\right)^2\right\}$$
$$\times\left[\left\{h(-\hat{z}) - \Phi(\hat{z})\right\}\frac{6}{24n}\hat{z}^4 - \frac{6}{24n}\phi(\hat{z})(\hat{z}^3 - \hat{z})\right] \quad (36)$$

Where, $\hat{z} = \sqrt{n}(r-1)/(r+1)$, $h(x) = 0, 1/2, 1$ when $x < 0, = 0, > 0$, respectively.

From (31) we can obtain another approximation formula for the tail probability of R_n as follows;

$$\widetilde{\widetilde{H}}_{n}^{LR}(r) = \Phi(-\sqrt{n}\hat{w}) + \phi(\sqrt{n}\hat{w})\frac{1}{\sqrt{n}}\left\{\frac{1}{\hat{\mu}} - \frac{1}{\hat{w}}\right\} \quad (37)$$

where, $\hat{\mu} = \tau(\Phi_n'')^{\frac{1}{2}}$ and $\hat{w} = \text{sgn}(\tau)\{\log\frac{(r+1)^2}{4r}\}^{\frac{1}{2}}$

Table 1 ~ Table 3 represent the numerical results of (34), (35), (36), (37) when n=2, 4, 16, with increasing r by 0.2.

The exact values of F-distributions are computed from IMSL. From the tables, we can see that the errors of approximation formulae of (16),(17) and (31) are small and that the results of (36) are more accurate than those of (35) in this case.

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Table 1. Exact probability from (34) and approximations $\tilde{\tilde{H}}_n^1$ from (35), $\tilde{\tilde{H}}_n^2$ from (36) and $\tilde{\tilde{H}}_n^{LR}$ from (37) for $P_r\{F(2,2) \geq r\}$ with increasing r by 0.2

r	Exact	$\tilde{\tilde{H}}_n^1$	$\tilde{\tilde{H}}_n^2$	$\tilde{\tilde{H}}_n^{LR}$
.1	0.9090909	0.9201847	0.9087645	0.9062693
.3	0.7692307	0.7882282	0.7679988	0.7663489
.5	0.6666667	0.6834431	0.6652367	0.6646699
.7	0.5882353	0.5987377	0.5872363	0.5871334
.9	0.5263158	0.5296686	0.5259858	0.5259482
1.0	0.5000000	0.5000000	0.5000000	0.5000000
1.1	0.4761904	0.4731556	0.4764897	0.4766421
1.3	0.4347826	0.4269092	0.4355608	0.4356063
1.5	0.4000000	0.3891510	0.4011196	0.4012423
1.7	0.3703703	0.3584602	0.3717733	0.3719602
1.9	0.3448275	0.3336820	0.3465602	0.3467019
2.1	0.3225806	0.3138670	0.3247937	0.3246891
2.3	0.3030303	0.2982295	0.3059649	0.3053323
2.5	0.2857143	0.2861151	0.2896828	0.2881766
2.7	0.2702702	0.2769766	0.2756359	0.2728652
2.9	0.2564102	0.2703549	0.2635686	0.2591152
3.1	0.2439024	0.2658640	0.2532657	0.2466982
3.3	0.2325581	0.2631778	0.2445425	0.2354289
3.5	0.2222222	0.2620202	0.2372370	0.2251545
3.7	0.2127659	0.2621581	0.2312074	0.2157484
3.9	0.2040816	0.2633923	0.2263262	0.2071046
4.1	0.1960784	0.2655538	0.2224799	0.1991337
4.3	0.1886792	0.2684984	0.2195667	0.1917598
4.5	0.1818182	0.2721027	0.2174943	0.1849180
4.7	0.1754386	0.2762612	0.2161795	0.1785525
4.9	0.1694915	0.2808834	0.2155469	0.1726150
5.1	0.1639344	0.2858917	0.2155288	0.1670637
5.3	0.1587301	0.2912192	0.2160628	0.1618618
5.5	0.1538461	0.2968087	0.2170936	0.1569774
5.7	0.1492537	0.3026108	0.2185703	0.1523821
5.9	0.1449275	0.3085827	0.2204467	0.1480507

Table 2. Exact probability from (34) and approximations $\tilde{\tilde{H}}_n^1$ from (35), $\tilde{\tilde{H}}_n^2$ from (36) and $\tilde{\tilde{H}}_n^{LR}$ from (37) for $P_r\{F(4,4) \geq r\}$ with increasing r by 0.2

r	Exact	$\tilde{\tilde{H}}_n^1$	$\tilde{\tilde{H}}_n^2$	$\tilde{\tilde{H}}_n^{LR}$
.1	0.9767092	0.9787889	0.9767641	0.9763958
.3	0.8648155	0.8732643	0.8647730	0.8641580
.5	0.7407408	0.7508543	0.7404818	0.7401797
.7	0.6309789	0.6382922	0.6307098	0.6306437
.9	0.5394372	0.5419201	0.5393354	0.5393090
1.0	0.5000000	0.5000000	0.5000000	0.5000000
1.1	0.4643127	0.4620604	0.4644052	0.4645124
1.3	0.4027287	0.3969803	0.4029525	0.4029841
1.5	0.3520000	0.3440123	0.3522837	0.3523745
1.7	0.3099121	0.3006226	0.3102054	0.3103792
1.9	0.2747139	0.2647627	0.2749883	0.2752488
2.1	0.2450405	0.2348481	0.2452818	0.2456237
2.3	0.2198292	0.2096659	0.2200327	0.2204462
2.5	0.1982507	0.1882858	0.1984163	0.1988901
2.7	0.1796537	0.1699897	0.1797838	0.1803066
2.9	0.1635227	0.1542181	0.1636209	0.1641822
3.1	0.1494465	0.1405312	0.1495169	0.1501075
3.3	0.1370949	0.1285795	0.1371412	0.1377532
3.5	0.1262003	0.1180836	0.1262261	0.1268529
3.7	0.1165445	0.1088174	0.1165529	0.1171892
3.9	0.1079482	0.1005970	0.1079420	0.1085832
4.1	0.1002631	0.9327122	0.1002447	0.1008872
4.3	0.0933656	0.0867153	0.0933371	0.0939778
4.5	0.0871525	0.0808255	0.0871156	0.0877522
4.7	0.0815365	0.0755146	0.0814927	0.0821235
4.9	0.0764440	0.0707094	0.0763946	0.0770179
5.1	0.0718121	0.0663478	0.0717582	0.0723729
5.3	0.0675872	0.0623778	0.0675297	0.0681348
5.5	0.0637232	0.0587517	0.0636627	0.0642578
5.7	0.0601803	0.0554330	0.0601176	0.0607019
5.9	0.0569238	0.0523874	0.0568595	0.0574329

Table 3. Exact probability from (34) and approximations $\tilde{\tilde{H}}_n^1$ from (35), $\tilde{\tilde{H}}_n^2$ from (36) and $\tilde{\tilde{H}}_n^{LR}$ from (37) for $P_r\{F(16,16) \geq r\}$ with increasing r by 0.2

r	Exact	$\tilde{\tilde{H}}_n^1$	$\tilde{\tilde{H}}_n^2$	$\tilde{\tilde{H}}_n^{LR}$
.1	0.9999833	0.9999839	0.9999834	0.9999833
.3	0.9894283	0.9897301	0.9894339	0.9894191
.5	0.9117686	0.9135296	0.9117789	0.9117389
.7	0.7581858	0.7608190	0.7581771	0.7581565
.9	0.5821548	0.5833845	0.5821451	0.5821324
1.0	0.5000000	0.5000000	0.5000000	0.5000000
1.1	0.4255825	0.4244589	0.4255913	0.4256431
1.3	0.3029717	0.3005737	0.3029850	0.3029969
1.5	0.2131033	0.2104668	0.2131089	0.2131343
1.7	0.1494310	0.1470498	0.1494277	0.1494636
1.9	0.1050326	0.1030616	0.1050237	0.1050637
2.1	0.0742443	0.0726825	0.0742328	0.0742720
2.3	0.0528793	0.0516696	0.0528675	0.0529034
2.5	0.0379884	0.0370616	0.0379773	0.0380087
2.7	0.0275411	0.0268343	0.0275312	0.0275580
2.9	0.0201534	0.0196146	0.0201450	0.0201674
3.1	0.0148844	0.0144729	0.0148774	0.0148959
3.3	0.0110930	0.0107776	0.0110872	0.0111025
3.5	0.0083404	0.0080976	0.0083357	0.0083482
3.7	0.0063243	0.0061363	0.0063204	0.0063306
3.9	0.0048348	0.0046885	0.0048316	0.0048400
4.1	0.0037250	0.0036105	0.0037224	0.0037293
4.3	0.0028914	0.0028014	0.0028893	0.0028950
4.5	0.0022604	0.0021891	0.0022587	0.0022633
4.7	0.0017791	0.0017224	0.0017777	0.0017816
4.9	0.0014094	0.0013641	0.0014083	0.0014115
5.1	0.0011234	0.0010870	0.0011225	0.0011252
5.3	0.0009008	0.0008713	0.0009000	0.0009022
5.5	0.0007264	0.0007024	0.0007257	0.0007275
5.7	0.0005888	0.0005693	0.0005883	0.0005898
5.9	0.0004798	0.0004638	0.0004793	0.0004806