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Asymptotics for Accelerated Life Test Models under Type II Censoring

Byung-Gu Park • Sang-Chul Yoon

Abstract Accelerated life testing(ALT) of products quickly yields information on life. In this paper, we investigate asymptotic normalities of maximum likelihood(ML) estimators of parameters for ALT model under Type II censored data using results of Bhattacharyya (1985). Further illustrations include the treatment of asymptotic of the exponential and Weibull regression models.

keywords: Accelerated Life Testing, Type II Censoring, Asymptotic Normality, Strong Consistency, Exponential and Weibull Regression Models.

1. Introduction

In many reliability fields, most of modern products and materials are designed to operate without failure for a long period. A few units will fail or degrade in the test of a practical length at use conditions. For this reason, ALT are used widely in manufacturing industries, to obtain time information quickly on the lifetime distribution of the life for products and materials. So units is tested at higher than normal level of stress involving high temperature, voltage, pressure, vibration, cycling rate, load, etc., to induce early failures.

Meeker and Nelson(1975) considered ML estimators of a percentile of an extreme value distribution at use condition. The percentile is a simple linear regression function of a stress, and scale parameter is a constant They gave the optimum plans for simultaneous testing with Type I censored data. Shaked and Singpurwalla(1982) considered a nonparametric version of ALT when there is a single cause of failure and the sample are not censored. Also, Viertl(1988), Nelson(1990), and Meeker and Escobar(1993) contain good references. Much of the previous works in ALT model area has focused on inference. Moreover, Bowmam and Shenton(1987) gave theory for the higher-order terms for greater accuracy of the asymptotic theory for the ML sampling distributions. Lu(1990) derived least square estimators of the model parameters and their asymptotic distributions. Nevertheless, always lags developed asymptotic theory

Byung-Gu Park is Professor, Department of Statistics, Kyungpook National University, Taegu, 702-701, Korea. Sang-Chul Yoon is Part-time Lecturer, Department of Statistics, Kyungpook National University, Taegu, 702-701, Korea. The present study was supported by the Basic Research Institute Program, Ministry of Education, 1996, Project No. BSRI-96.

and methods for ALT model.

In this paper, we study asymptotic normality of ML estimators for the parameters of ALT models under Type II censoring, assuming that the log-lifetime is a linear function of stresses.

In Section 2, we review asymptotic properties of class of functions that arise in the context of estimating parameters from Type II censored data.

In Section 3,we derive ML estimatбrs of parameters for ALT models under Type II censored data, and investigate their asymptotic normalities using results of Bhattacharyya(1985).

In Sections 4, we deal with two examples to illustrate for results of Section 3 under exponential and Weibull regression models.

2. Review of the Principal tools for Maximum Likelihood Estimators

In a life testing, we assume the lifetimes $T_1, T_2, \cdots, T_n$ are independently and identically distributed(iid) random variables having a common continuous distribution. Let $Y_1 \le Y_2 \le \cdots \le Y_n$ be the order statistics of $T_1, T_2, \cdots, T_n$. The most common situation is Type II censoring, which permits the first $r(<n)$ order statistics to be observed.

Let $f(t,\underline{\beta})$ and $F(t,\underline{\beta})$ denote the probability density function(pdf) and the distribution function(df) of the lifetime t, respectively. For a possibly vector-valued parameter $\underline{\beta} \in \Omega \subset R^l$, the log-likelihood function for Type II censored data is

$$\mathrm{L}_n(\underline{\beta}) = \log\frac{n!}{(n-r)!} + \sum_{i=1}^{r} \log f(Y_i,\ \underline{\beta}) + (n-r)\log \bar{F}(Y_i,\ \underline{\beta}), \tag{2.1}$$

where $\bar{F} = 1 - F$. Throughout, all limits are taken as $n \to \infty$; where $p \in (0,1)$ is fixed, $r = [np]$. It is assumed that $f(t)$ is continuous at ς, the p-th quantile of the failure time distribution, and that $f_i(\varsigma_i) > 0$. The notation $\xrightarrow{d} N_k(\underline{\mu}, \Sigma)$ will be used for convergence in distribution to a k-variate normal with mean $\underline{\mu}$ and covariance matrix Σ. Vectors are meant to be column vectors and $*$ denote their transpose.

By using (2.1), ML estimations under the ALT models is widely discussed in the reliability problem. In general, it is difficult for us to obtain the asymptotics because the M.L. estimators, based on Type II censored data, are neither independent nor identically distributed. From the form of the log-likelihood (2.1), Bhattacharyya(1985) suggested the random vector of the form

$$\underline{\mathrm{T}}_n(\underline{\beta}) = n^{-1}\left[\sum_{i=1}^{r} \underline{g}(Y_i,\ \underline{\beta}) + (n-r)\underline{h}(Y_r,\ \underline{\beta})\right], \tag{2.2}$$

where $\underline{g}$ and $\underline{h}$ are functions on $\mathfrak{I} \times \Omega \to R^l$ and $\mathfrak{I}$ denotes the sample space of T_1.

Assume that the true value $\underline{\beta}_0$ is an interior point of $\Omega \subset R^k$. Let $f(t)$ stand for $f(t,\underline{\beta}_0)$,

$\mathrm{g}(t)$ for $\mathrm{g}(t,\underline{\beta}_0)$, and

$$\underline{\mathrm{T}}_n = \underline{\mathrm{T}}_n(\underline{\beta}_0) = n^{-1}\left[\sum_{i=1}^{r}\underline{\mathrm{g}}(Y_i) + (n-r)\underline{\mathrm{h}}(Y_r)\right]. \tag{2.3}$$

Then Lemma 2.1 and Lemma 2.2 give the asymptotic properties of T_n defined in (2.3).

Lemma 2.1 (Asymptotic Normality, Bhattacharyya(1985)). Let g_α and h_α denote the αth coordinate of $\underline{\mathrm{g}}$ and $\underline{\mathrm{h}}$ for $\alpha = 1,2,\cdots,l$, respectively, and assume that

(1) $\mathrm{h}_\alpha'(t) = d\mathrm{h}_\alpha(t)/dt$ exists at $t = \varsigma$,
(2) $\mathrm{h}_\alpha(t)$ is continuous at ς,
and
(3) $\int_{\infty}^{\varsigma} \mathrm{g}_\alpha^2(t)\mathrm{f}(t)\,dt < \infty$.

Then it follows that

$$\sqrt{n}\left(\underline{\mathrm{T}}_n - \underline{\mu}\right) \xrightarrow{d} \mathrm{N}_l(\underline{0}, \Sigma),$$

where

$$\underline{\mu} = \int_{\infty}^{\varsigma} \underline{\mathrm{g}}(t)\,\mathrm{f}(t)\,dt + \mathrm{q}\,\underline{\mathrm{h}}(\varsigma),$$
$$\Sigma = \underline{\tau} + \mathrm{pq}\,\mathrm{f}^{-2}(\varsigma)\underline{\mathrm{b}}\underline{\mathrm{b}}^*,$$
$$\underline{\tau} = \int_{\infty}^{\varsigma} \underline{\mathrm{g}}(t)\,\underline{\mathrm{g}}^*(t)\,\mathrm{f}(t)\,dt - \mathrm{p}^{-1}\left(\int_{\infty}^{\varsigma} \underline{\mathrm{g}}(t)\,\mathrm{f}(t)\,dt\right)\left(\int_{\infty}^{\varsigma} \underline{\mathrm{g}}(t)\,\mathrm{f}(t)\,dt\right)^*,$$

and

$$\underline{\mathrm{b}} = \mathrm{f}(\varsigma)\,\underline{\mathrm{g}}(\varsigma) - \mathrm{p}^{-1}\int_{\infty}^{\varsigma} \underline{\mathrm{g}}(t)\,\mathrm{f}(t)\,dt + \mathrm{q}\,\underline{\mathrm{h}}'(\varsigma). \tag{2.4}$$

Lemma 2.2 (Strong Consistency, Bhattacharyya(1985)). Assume the followings for a compact neighborhood B of $\underline{\beta}_0$.

(1) $\mathrm{g}(t,\underline{\beta})$ is continuous at $\underline{\beta} \in \mathrm{B}$ for every t,
(2) For $\underline{\beta} \in \mathrm{B}$ and all t, $|\mathrm{g}(t,\underline{\beta})| \le \mathrm{g}_0(t)$ such that $\int_{\infty}^{\varsigma} \mathrm{g}_\alpha^2(t)\mathrm{f}(t)\,dt < \infty$,
(3) $\mathrm{g}(t,\underline{\beta})$ is continuous on $[\varsigma-\varepsilon_1, \varsigma+\varepsilon_1]\times\mathrm{B}$ for some $\varepsilon_1 > 0$.

Then it follows that

$$\sup_{\underline{\beta}\in\mathrm{B}}\left|\underline{\mathrm{T}}_n(\underline{\beta}) - \underline{\mu}(\underline{\beta})\right| \xrightarrow{a.s.} 0, \tag{2.5}$$

where $\underline{\mu}(\underline{\beta}) = \int_{\infty}^{\infty} \mathrm{g}(t,\underline{\beta})\,\mathrm{f}(t)\,dt + \mathrm{q}\,\mathrm{h}(\varsigma,\underline{\beta})$

3. Asymptotic Properties for Accelerated Life Tests

In this section, we will study the asymptotic properties of M.L. estimators for ALT models under Type II censored data.

Assuming $\phi(t,\underline{\beta}) = \log f(t,\underline{\beta})$ and $\rho(t,\underline{\beta}) = \log F(t,\underline{\beta})$. We can derive ALT model. The log-likelihood function of (2.1) is represented as

$$\mathrm{L}_n(\underline{\beta}) = \log\frac{n!}{(n-r)!} + \sum_{i=1}^{r}\phi(Y_i,\underline{\beta}) + (n-r)\rho(Y_r,\underline{\beta}).$$

The first derivative of $\mathrm{L}_n(\underline{\beta})$ is given as

$$\mathrm{L}_n^{'}(\underline{\beta}) = \frac{\partial \mathrm{L}_n(\underline{\beta})}{\partial \underline{\beta}} = \sum_{i=1}^{r}\underline{\phi}^{'}(Y_i,\underline{\beta}) + (n-r)\underline{\rho}^{'}(Y_r,\underline{\beta}), \tag{3.1}$$

where $\underline{\phi}^{'}(Y_i,\underline{\beta}) = (\partial/\partial\underline{\beta})\,\phi(x,\underline{\beta})$ and $\underline{\rho}^{'}(Y_i,\underline{\beta}) = (\partial/\partial\underline{\beta})\rho(x,\underline{\beta})$,

and the second derivative of $\mathrm{L}_n(\underline{\beta})$ is given as

$$\mathrm{L}_n^{''}(\underline{\beta}) = \frac{\partial^2 \mathrm{L}_n(\underline{\beta})}{\partial \underline{\beta}_a \partial \underline{\beta}_e} = \sum_{i=1}^{r}\underline{\phi}^{''}(Y_i,\underline{\beta}) + (n-r)\underline{\rho}^{''}(Y_r,\underline{\beta}), \tag{3.2}$$

where $\underline{\phi}^{''}(Y_i,\underline{\beta}) = (\partial^2/\partial\underline{\beta}_a\partial\underline{\beta}_e)\,\phi(x,\underline{\beta})$ and $\underline{\rho}^{''}(Y_i,\underline{\beta}) = (\partial^2/\partial\underline{\beta}_a\partial\underline{\beta}_e)\rho(x,\underline{\beta})$.

Therefore the likelihood equation is

$$\mathrm{L}_n^{'}(\underline{\beta}) = \sum_{i=1}^{r}\underline{\phi}^{'}(Y_i,\underline{\beta}) + (n-r)\underline{\rho}^{'}(Y_r,\underline{\beta}) = 0.$$

Let $\hat{\underline{\beta}}_n$ be a function of the data from Type II censored under ALT model. Then the following lemmas and theorems are used in finding the asymptotic distribution of $\sqrt{n}(\hat{\underline{\beta}}_n - \underline{\beta}_0)$ of Theorem 3.3.

Lemma 3.1 Assume that $\hat{\underline{\beta}}_n$ is a strong consistent sequence of estimators, and $\mathrm{L}^{'}$ is differentiable at $\underline{\beta}_0$. Then

$$\underline{\Lambda}(\hat{\underline{\beta}}_n) \xrightarrow{a.s.} 0,$$

where $\underline{\Lambda}(\hat{\underline{\beta}}_n) = \mathrm{L}^{'}(\hat{\underline{\beta}}_n) - \mathrm{L}^{'}(\underline{\beta}_0) - \mathrm{L}^{''}(\hat{\underline{\beta}}_n - \underline{\beta}_0)$.

Proof. It is easily proved by Taylor expansion .

Theorem 3.1 Under the conditions of Lemma 2.1, we assume the derivative $(\partial/\partial\underline{\beta})\int_{-\infty}^{\infty} f(x,\underline{\beta})\,dx$ can be carried within the integral. Then

$$\sqrt{n}(n^{-1}\,\mathrm{L}^{'}(\underline{\beta}_0)) \xrightarrow{d} \mathrm{N}_k(\,\underline{0},\ \Sigma\,),$$

where $\mathrm{S} = \int_{\infty}^{\varsigma} \underline{\mathrm{f}}'(t)\, \underline{\mathrm{f}}'^{*}(t)\, \mathrm{f}(t)\, dt + \mathrm{q}^{-1}\left(\int_{\infty}^{\varsigma} \underline{\mathrm{f}}'(t)\, \mathrm{f}(t)\, dt\right)\left(\int_{\infty}^{\varsigma} \underline{\mathrm{f}}'(t)\, \mathrm{f}(t)\, dt\right)^{*}.$ (3.3)

Proof. Using Lemma 2.1, we can obtain the limiting distribution of $\sqrt{n}(n^{-1}\mathrm{L}'(\underline{\beta}_0))$. Now, to obtain explicit expressions for mean and covariance of $\sqrt{n}(\hat{\underline{\beta}}_n - \underline{\beta}_0)$, it can be rewritten by $\underline{\mathrm{h}}$

$$\underline{\mathrm{h}}(t,\underline{\beta}) = \underline{\rho}'(t,\underline{\beta}) = \frac{\partial}{\partial \underline{\beta}} \log \bar{\mathrm{F}}(t,\underline{\beta}) = -(\bar{\mathrm{F}}(t,\underline{\beta}))^{-1} \int_{\infty}^{t} \frac{\partial}{\partial \underline{\beta}} \mathrm{f}(t,\underline{\beta})\, dt. \tag{3.4}$$

By the previous definitions of ϕ and $\underline{\phi}'$ we obtain the followings

$$\underline{\mathrm{g}}(t,\underline{\beta}) = \underline{\phi}'(t,\underline{\beta}) = \frac{\partial}{\partial \underline{\beta}} \log \mathrm{f}(t,\underline{\beta}) = -(\mathrm{f}(t,\underline{\beta}))^{-1} \frac{\partial}{\partial \underline{\beta}} \mathrm{f}(t,\underline{\beta}), \tag{3.5}$$

and then (3.5) is give as follows

$$\frac{\partial}{\partial \underline{\beta}} \mathrm{f}(t,\underline{\beta}) = \underline{\mathrm{g}}(t,\underline{\beta})\, \mathrm{f}(t,\underline{\beta}) = \underline{\phi}'(t,\underline{\beta})\, \mathrm{f}(t,\underline{\beta}). \tag{3.6}$$

Therefore

$$\underline{\mathrm{h}}(t,\underline{\beta}) = -(\bar{\mathrm{F}}(t,\underline{\beta}))^{-1} \int_{\infty}^{t} \underline{\mathrm{g}}(x,\underline{\beta})\, \mathrm{f}(x,\underline{\beta})\, dx \tag{3.7}$$

and then the $\underline{\mathrm{h}}'(t)$ with respect to $t = \varsigma$ is computed as

$$\underline{\mathrm{h}}'(t) = (\bar{\mathrm{F}}(t))^{-1}\left[-\, \underline{\mathrm{g}}(t)\, \mathrm{f}(t) + \underline{\mathrm{h}}(t)\, \mathrm{f}(t)\right]. \tag{3.8}$$

Therefore it follows that the equation (3.8) is obtained as

$$\underline{\mathrm{h}}(\varsigma) = -\,\mathrm{q}^{-1} \int_{\infty}^{\varsigma} \underline{\mathrm{g}}(x)\, \mathrm{f}(x)\, dx, \tag{3.9}$$

and

$$\underline{\mathrm{h}}'(\varsigma) = \mathrm{q}^{-1}\left[-\, \underline{\mathrm{g}}(\varsigma)\, \mathrm{f}(\varsigma) + \underline{\mathrm{h}}(\varsigma)\, \mathrm{f}(\varsigma)\right], \tag{3.10}$$

where $\mathrm{q} = 1 - \mathrm{F}(\varsigma)$. Using these results for $t = \varsigma$, and using (3.9) and (3.10), we caculated $\underline{\mu}, \underline{\mathrm{b}}$ and Σ as follows.

$$\underline{\mu} = 0, \quad \underline{\mathrm{b}} = -(\mathrm{pq})^{-1}\, \mathrm{f}(\varsigma) \int_{\infty}^{\varsigma} \underline{\phi}'(t)\, \mathrm{f}(t)\, dt$$

and

$$\Sigma = \int_{\infty}^{\varsigma} \underline{\phi}'(t)\, \underline{\phi}^{*}(t)\, \mathrm{f}(t)\, dt + \mathrm{q}^{-1}\left(\int_{\infty}^{\varsigma} \underline{\phi}'(t)(t)\, \mathrm{f}(t)\, dt\right)\left(\int_{\infty}^{\varsigma} \underline{\phi}'(t)(t)\, \mathrm{f}(t)\, dt\right)^{*}.$$

Finally, Lemma 2.1 implies

$$\sqrt{n}(n^{-1}\,\mathrm{L}'(\underline{\beta}_0)) \xrightarrow{d} \mathrm{N}_k(\underline{0}, \Sigma), \tag{3.11}$$

and then this completes the proof.

Theorem 3.2 Under the conditions of Lemma 2.2, assume $\hat{\underline{\beta}}_n$ is a strong consistent sequence of estimators and the derivative $(\partial/\partial\underline{\beta})\int_{-\infty}\mathrm{f}(x,\underline{\beta})\,dx$ can be carried within the integral. Then

$$-\,n^{-1}\,\mathrm{L}''(\underline{\beta}_0) \xrightarrow{a.s.} \Sigma, \tag{3.12}$$

where Σ is given in (3.3).

Proof. Since $\mathrm{T}_n(\underline{\beta})$ in (2.2) is given as

$$\mathrm{T}_n(\underline{\beta}) = n^{-1}\left[\sum_{i=1}^{r}\phi''(t,\underline{\beta}) + (n-r)\rho''(t,\underline{\beta})\right], \tag{3.13}$$

Lemma 2.2 is represented as follows

$$\sup_{\underline{\beta}\in\mathrm{B}}\left|\,\mathrm{T}_n(\underline{\beta}) - \mu(\underline{\beta})\,\right| \xrightarrow{a.s.} 0, \tag{3.14}$$

where $\mu(\underline{\beta}) = \int_{-\infty}^{\varsigma}\phi''(t,\underline{\beta})\mathrm{f}(t)\,dt + \mathrm{q}\,\rho''(\varsigma,\underline{\beta}).$ (3.15)

Now it will be shown that after making certain assumptions $\mu(\underline{\beta}) = \Sigma$ is as given in (3.3). To do this, $\int_{-\infty}^{\varsigma}\mathrm{f}(t,\underline{\beta})\,dt + \bar{\mathrm{F}}(\varsigma,\underline{\beta}) = 1$ is differentiated twice with respect to $\underline{\beta}$. Upon differentiating once, we obtain

$$\frac{\partial}{\partial\underline{\beta}}\int_{-\infty}^{\varsigma}\mathrm{f}(t,\underline{\beta})\,dt + \frac{\partial}{\partial\underline{\beta}}\bar{\mathrm{F}}(\varsigma,\underline{\beta}) = \underline{0}. \tag{3.16}$$

By using (3.4) and (3.6), (3.16) leads to

$$\int_{-\infty}^{\varsigma}\underline{\phi}'(t,\underline{\beta})\,\mathrm{f}(t,\underline{\beta})\,dt + \underline{\rho}'(\varsigma,\underline{\beta})\bar{\mathrm{F}}(\varsigma,\underline{\beta}) = \underline{0}. \tag{3.17}$$

Differentiating (3.17) with respect to $\underline{\beta}$

$$\frac{\partial}{\partial\underline{\beta}}\int_{-\infty}^{\varsigma}\underline{\phi}'(t,\underline{\beta})\,\mathrm{f}(t,\underline{\beta})\,dt = \int_{-\infty}^{\varsigma}\frac{\partial}{\partial\underline{\beta}}\underline{\phi}'(t,\underline{\beta})\,\mathrm{f}(t,\underline{\beta})\,dt,$$

we obtain

$$\begin{aligned}&\int_{-\infty}^{\varsigma}\underline{\phi}''(t,\underline{\beta})\,\mathrm{f}(t,\underline{\beta})\,dt + \Big[\int_{-\infty}^{\varsigma}\underline{\phi}'(t,\underline{\beta})\frac{\partial}{\partial\underline{\beta}}\mathrm{f}(t,\underline{\beta})\,dt\Big]^{*}\\ &+ \underline{\rho}'(\varsigma,\underline{\beta})\,\bar{\mathrm{F}}(\varsigma,\underline{\beta}) + \underline{\rho}'(\varsigma,\underline{\beta})\,(\underline{\rho}'(\varsigma,\underline{\beta}))^{*}\,\bar{\mathrm{F}}(\varsigma,\underline{\beta}) = 0.\end{aligned} \tag{3.18}$$

Using (3.6), (3.18) and putting $\underline{\beta}=\underline{\beta}_0$, it follows that

$$-\mu(\underline{\beta}_0)=\int_{-\infty}^{\varsigma}\underline{\phi}'(t)(\underline{\phi}'(t))^{*}\,\mathrm{f}(t)\,dt+\mathrm{q}\,\underline{\rho}'(\varsigma)(\underline{\rho}'(t))^{*}, \tag{3.19}$$

where $\mathrm{q}=\bar{\mathrm{F}}(\varsigma,\underline{\beta})$. From (3.4) and (3.18) we obtain as follows.

$$\underline{\rho}'(t,\underline{\beta})=-((\bar{\mathrm{F}}(t,\underline{\beta}))^{-1}\int_{-\infty}^{t}\underline{\phi}'(x,\underline{\beta})\mathrm{f}(x,\underline{\beta})\,dx \tag{3.20}$$

and

$$\underline{\rho}'(\varsigma)=-\mathrm{q}^{-1}\int_{-\infty}^{\varsigma}\underline{\phi}'(x)\mathrm{f}(x)\,dx. \tag{3.21}$$

Substituting (3.20) into (3.21), we obtain

$$-\mu(\underline{\beta}_0)=\int_{-\infty}^{\varsigma}\underline{\phi}'(\underline{\phi}')^{*}\,\mathrm{f}\,dt+\mathrm{q}^{-1}[\int_{-\infty}^{\varsigma}\underline{\phi}'\,\mathrm{f}\,dt][\int_{-\infty}^{\varsigma}\underline{\phi}'\,\mathrm{f}\,dt]^{*}=\Sigma. \tag{3.22}$$

Hence, from (3.13), (3.15) and (3.22),

$$-n^{-1}\,\mathrm{L}''(\underline{\beta}_0)\xrightarrow{a.s.}\Sigma, \tag{3.23}$$

and then this proof completes.

Using Lemma 3.1, Theorem 3.1 and Theorem 3.2, we establish the asymptotic normality of $\sqrt{n}(\hat{\underline{\beta}}_n-\underline{\beta}_0)$ for ALT models under Type II censored data.

Theorem 3.3 Under the conditions of Lemma 2.1 and Lemma 2.2, assume

(1) $\hat{\underline{\beta}}_n$ is strong consistent sequence of estimators,

(2) the derivative $(\partial/\partial\underline{\beta})\int_{-\infty}^{\infty}\mathrm{f}(x,\underline{\beta})\,dx$ can be carried within the integral.

Then

$$\sqrt{n}\left(\hat{\underline{\beta}}_n-\underline{\beta}_0\right)\xrightarrow{d}\mathrm{N}_k(\underline{0},\ \Sigma^{-1}),$$

where Σ is defined in (3.3).

Proof. By Lemma 3.1,

$$-\underline{\Delta}(\hat{\underline{\beta}}_n)=\mathrm{L}'(\underline{\beta}_0)+\mathrm{L}''(\underline{\beta}_0)(\hat{\underline{\beta}}_n-\underline{\beta}_0),$$

and multiplying both sides by $n^{-(1/2)}$ gives

$$-n^{-\frac{1}{2}}\underline{\Delta}(\hat{\underline{\beta}}_n)=\sqrt{n}\,(n^{-1}\,\mathrm{L}'(\underline{\beta}_0))-(-n^{-1}\,\mathrm{L}''(\underline{\beta}_0))(\sqrt{n}(\hat{\underline{\beta}}_n-\underline{\beta}_0)).$$

Also, from Lemma 3.1, we can obtain easily

$$-\frac{1}{\sqrt{n}}\underline{\Lambda}(\hat{\underline{\beta}}_n) \xrightarrow{a.s.} \underline{0}. \tag{3.24}$$

Using (3.24), and Theorem 3.1 and Theorem 3.2, one can obtain

$$\sqrt{n}\,(\hat{\underline{\beta}}_n - \underline{\beta}_0) \xrightarrow{d} N_k(\underline{0},\ \Sigma^{-1}),$$

by the Slutsky's theorem.

4. Examples

In this section, we consider the applications in two ALT models: the exponential and the Weibull regression models.

Weibull Regression Model

As an application for the limiting distribution of $\sqrt{n}(\hat{\underline{\beta}}_n - \underline{\beta}_0)$, we consider the pdf $f(t,\underline{\beta})$ of lifetime t for a unit under stress $\underline{z}$ given as

$$f(t,\underline{\beta}) = \left(\frac{\gamma}{\theta}\right)\left(\frac{t}{\theta}\right)^{\gamma-1}\exp\left[-\left(\frac{t}{\theta}\right)^{\gamma}\right],$$

where $\theta = \exp(\underline{z}^*\underline{\beta})$, $\underline{z} = (z_1, z_2, \cdots, z_p)^*$ and $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_p)^*$. Since

$$\phi(t,\underline{\beta}) = \log f(t,\underline{\beta}) = \log\gamma - \underline{z}^*\underline{\beta} + (\gamma-1)\log t - (\gamma-1)\underline{z}^*\underline{\beta} - \left(t\exp-(\underline{z}^*\underline{\beta})\right)^{\gamma},$$

it follows that

$$\underline{\phi}'(t,\underline{\beta}) = \frac{\partial}{\partial\underline{\beta}}\phi(t,\underline{\beta}) = \gamma\left(\frac{t}{\theta}\right)^{\gamma-1}\underline{z}\,.$$

Now Σ for this model will be found. First, we will compute $\int_{-\infty}^{\varsigma}\underline{\phi}^{*} f\,dt$ as follows

$$\int_{\infty}^{\varsigma}\underline{\phi}' f\,dt = -\left(\frac{\varsigma}{\theta}\right)^{\gamma}\gamma\, e^{-\left(\frac{\varsigma}{\theta}\right)}\underline{z}\,.$$

Let $p = \int_{-\infty}^{\varsigma} f(t,\underline{\beta})dt$. The p is represented as $\log(-\log(1-p)) = \gamma\log\varsigma - \gamma\log\theta$. Then we obtain

$$\int_{\infty}^{\varsigma}\underline{\phi}^{*} f\,dt = \gamma\left[(1-p)\log(1-p)\right]\underline{z}$$

and

$$\left(\int_{\infty}^{\varsigma}\underline{\phi}' f\,dt\right)\left(\int_{\infty}^{\varsigma}\underline{\phi}' f\,dt\right)^{*} = \gamma^2\,(q\log q)^2\,\underline{z}\underline{z}^{*}. \tag{4.1}$$

Since

$$\underline{\phi}'(\underline{\phi}')^{*} = \gamma^{2}\left[\left(\frac{t}{\theta}\right)^{\gamma} - 1\right]^{2} \underline{z}\,\underline{z}^{*},$$

it follows that

$$\left(\int_{-\infty}^{\infty} \underline{\phi}' \underline{\phi}'^{*} \, \mathrm{f}\, dt\right) = \gamma^{2}\,(\mathrm{p} - \mathrm{q}\log^{2}\mathrm{q})\,\underline{z}\underline{z}^{*}. \tag{4.2}$$

From (4.1) and (4.2), we obtain

$$\Sigma = \gamma^{2}\,\mathrm{p}\,\underline{z}\underline{z}^{*}.$$

Hence we obtain the asymptotic distribution of $\sqrt{n}(\hat{\underline{\beta}}_n - \underline{\beta}_0)$ for the Weibull regression model

$$\sqrt{n}\left(\hat{\underline{\beta}}_n - \underline{\beta}_0\right) \xrightarrow{d} \mathrm{N}_p\left(\underline{0},\, \left(\gamma^{2}\,\mathrm{p}\underline{z}\underline{z}^{*}\right)^{-1}\right).$$

Exponential Regression Model

This model reduce to the Weibull regression model when $\gamma = 1$. So, we consider the pdf $\mathrm{f}(t,\underline{\beta})$ of lifetime t a unit under stress $\underline{z}$ given as

$$\mathrm{f}(t,\underline{\beta}) = \theta^{-1}\exp\left(-\frac{t}{\theta}\right), \quad 0 < t < \infty,$$

where $\theta = \exp(\underline{z}^{*}\underline{\beta})$, $\underline{z} = (z_1, z_2, \cdots, z_p)^{*}$ and $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_p)^{*}$. Therefore we obtain the limiting distribution of $\sqrt{n}(\hat{\underline{\beta}}_n - \underline{\beta}_0)$ for the exponential regression model

$$\sqrt{n}\left(\hat{\underline{\beta}}_n - \underline{\beta}_0\right) \xrightarrow{d} \mathrm{N}_p\left(\underline{0},\, \left(\mathrm{p}\underline{z}\underline{z}^{*}\right)^{-1}\right).$$

Remark. Exponential-Arrhenius Model

The Arrhenius model is a special case of the exponential regression model with $\mathrm{p} = 2$ where $\underline{z} = (1, x)^{*}$ and $\underline{\beta} = (\beta_1, \beta_2)^{*}$. Then, it is easily obtained that the limiting distribution of $\hat{\underline{\beta}}$ for the Arrhenius model is given as

$$\sqrt{n}\left(\hat{\underline{\beta}}_n - \underline{\beta}_0\right) \xrightarrow{d} \mathrm{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix},\, \begin{pmatrix}\mathrm{p}^{-1} & 0\\ 0 & (\mathrm{p}x^{2})^{-1}\end{pmatrix}\right).$$

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