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Some Partial Orders Describing Positive Aging

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Abstract The concepy of positive aging describles the adverse effects of age on the lifetime of units. Various aspects of this concepts are described in terms of conditional probability distribution of residual life times, failure rates, equilibrium distributions, etc. In this paper we will consider some partial ordering relations of life distribution under residual life functions and equilibrium distributions.

Key Words : Life distribution, Residual Life Distribution, Partial Orderings, Equilibrium Distribution, Positive Ageing classes.

1. Introduction

By the aging of a mechanical unit, component, or some other physical or biological systems, we mean the phenomenon by which an older system has ashorter remaining lifetime, in some stochastic sense, than a newer or younger one.

Suppose that X and Y are nonnegative absolutely continuous random variables with probability density functions $f(x)$ and $g(x)$, respectively. Let F and G be the cummulative distribution functions of X and Y , and $\overline{F}(x)=1-F(x)$ and $\overline{G}(x)=1-G(x)$ be the corresponding survival functions. Partial orderings, namely, $s-FR$ ordering(likelihood ratio ordering, failure rate ordering, mean residual life ordering), $s-ST$ ordering(weak likelihood ratio drdering, stochastic ordering, harmonic average mean life ordering), $s-SFR$ ordering(expectation ordering and initial failure rate ordering) between two random variables X and $Y$are known in the literature

Deshpande, Kochar and Singh(1986) introduced various aspects of this

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concept which are described in terms of conditional probability distributions of residual life times, failure rates, equilibrium distributions. Gupta(1987) studied how the ageing properties IFR, NBU, NBUE and DMRL of the original distribution were transformed into the ageing properties of the distribution of the residual life. Kochar and Wiens(1987) defined new partial orderings of life distributions and studied the relationships of some partial orderings. Singh(1989) defined two new partial orderings and discussed relevance of these partial orderings for comparing life of a new unit with residual lifeof a used unit.
Deshpande,Singh,Bagai and Jain(1990) investigated partial orders relations with existing partial orders of probability distributions for describing the phenomenon of ageing.
Gupta and Kirmani(1987) studied the relationship between the weighted distributions and parent distributions in the context of reliability and life testing. Fagiuoli and Pellerey(1993) introduced new concepts of partial stochastic ordering and studied relations among them and the classical partial orderings. Choi, Cho and Kim(1995) studied order relations among classical partial ordering .

The residual life of a component of age t and the equilibrium distribution are of great interest in actuarial studies, survival analysis and reliability. So, in Section 2, we shall consider some partial order relations of life distribution under the residual life function. In Section 3, we shall consider some partial order relations of life distribution under the equilibrium distribution.

2. Relations among Some Partial Orderings Under Residual Life Distributions

First of all, we briefly discuss below some of the well-known partial orderings(see Fagiuoli and Pellerey(1993)).

Given an absolutely continuous nonnegative random variable X, we denote $\overline{T_0}(x) = f(x)$ and $\overline{T_s}(x) = \dfrac{\int_x^\infty \overline{T_{s-1}}(\mu)d\mu}{\mu_{s-1}}$, for $s \geq 1$, where $\mu_s(x) = \int_0^\infty \overline{T_s}(\mu)d\mu$.

Also define

$$r_{T_s}(x) = \frac{\overline{T_{s-1}(x)}}{\int_x^{\infty} \overline{T_{s-1}}(\mu)d\mu} = \frac{-\frac{d}{dx}\overline{T}_s(x)}{\overline{T}_s(x)}, \text{ for } s \geq 1$$ and

$$r_{T_0}(x) = \frac{f'(x)}{f(x)} \text{ when } f'(x) \text{ exists.}$$

We use $\overline{U}_s(x)$, $r_{U_s}(x)$ and γ_s correponding to $\overline{T}_s(x)$, $r_{T_s}(x)$ and μ_s for the random variables Y.

Definition 1. X is said to be larger than Y in s-FR ordering, written as $X \overset{s-FR}{\geq} Y$, if $\frac{\overline{T}_s(x)}{\overline{U}_s(x)}$ is nondecreasing in $x \geq 0$.

Definition 2. X is said to be larger than Y in s-ST ordering, written as $X \overset{s-ST}{\geq} Y$, if $\frac{\overline{T}_s(x)}{\overline{U}_s(x)} \geq \frac{\overline{T}_s(0)}{\overline{U}_s(0)}$ for all $x \geq 0$.

Definition 3. X is said to be larger than Y in s-SFR ordering, written as $X \overset{s-SFR}{\geq} Y$, if $r_{T_s}(0) \leq r_{U_s}(0)$.

Definition 4. X is said to be a s-IFR if $r_{T_s}(x)$ is nondecreasing in $x \geq 0$.

Theorem 2.1. X be nonnegative s-IFR random variables, and let $a \leq 1$ be positive constant. Then $aX \overset{s-FR}{\leq} X$.

Proof. It is easy to verify that the functions $r_{U_s}(x)$ of aX is given by $\frac{1}{a} r_{T_s}(x/a)$, for all t, where $r_{T_s}(x)$ is the failure function of s-order equilibrium distribution of X. Now,

$r_{U_s}(x) = \frac{1}{a} r_{T_s}(x/a) \geq r_{T_s}(x/a) \geq r_{T_s}(x)$, for all x, where the first inequality follows from $a \in [0, 1]$ and the second inequality follows from the assuption that X is s-IFR.

Note that if X is the life time of a device then $[X - t | X > t]$ is the residual life of such a device with age t, and has the survival function $\overline{F}_t(x) = P[X > x+t | X > t] = \overline{F}(x+t)/\overline{F}(t)$.

Lemma 2.2.(Fagiuoli and Pellerey(1993)) Let X have distribution F and X_t have distribution G. With T_s and U_s we will mean the equilibrium distribution of order s of X and X_t, respectively.

$$\overline{U}_s(x) = \frac{\overline{T}_s(x+t)}{\overline{T}_s(t)}$$, for all $s \in N^+$, where $N^+ = \{x | x \text{ is natural number}\}$.

We note that $$r_{U_s}(x) = \frac{-\frac{d}{dx}\overline{U}_s(x)}{\overline{U}_s(x)} = \frac{-\frac{d}{dx}\overline{T}_s(x+t)}{\overline{T}_s(x+t)} = r_{T_s}(x+t).$$

Many partial orders are utilized for making comparsions between probability distributions of residual life times at different ages in order to describe positive ageing. Hence, we distinguish between two types of positive ageing.

i) Younger Better than Older (YBO) type ageing wherein the effect of ageing is progressive and the unit deteriorates monotonically, in some sense, with increasing age, t i.e., the probability distributions of X_t, $t > 0$ are seen to be ordered monotonically in the sense of a suitable order.

ii) New Better than Used (NBU) type ageing wherein the comparison is only between a new unit (i.e., of age 0) and a used unit(i.e., of age t>0). Positive ageing is asserted if the distributions of X_t, $t > 0$ can be ordered with respect to the distribution of X, the life time of a new unit. We observe that both the YBO and NBU comparisons in terms of some specific partial order lead to the same class of distributions.

We know that $X \overset{s-FR}{\geq} X_t$, $t > 0$ if and only if X is s-IFR.

Theorem 2.3: The following conditions are equivalent.

1. X is s-IFR.
2. $X \overset{s-FR}{\geq} X_t$, $t > 0$

3. $X_{t_1} \overset{s-FR}{\geq} X_{t_2}$, for all $o < t_1 \leq t_2$.
4. $X_{t_1} \overset{s-ST}{\geq} X_{t_2}$, for all $o < t_1 \leq t_2$.
5. $X_{t_1} \overset{s-SFR}{\geq} X_{t_2}$, for all $o < t_1 \leq t_2$.

Proof. Let X have distribution F and X_t have distribution F_t . With T_s and U_s we will mean the equilibrium distribution of order s of X and X_t , respectively, and also let V_s and W_s we will mean the equilibrium distribution of order s of X_{t_1} and X_{t_2}, respectively.

$$X_{t_1} \overset{s-ST}{\geq} X_{t_2} ,\ 0 < t_1 \leq t_2 \Leftrightarrow \overline{V}_s(x) \geq \overline{W}_s(x),\ x \geq 0$$

$$\Leftrightarrow \frac{\overline{T}_s(t_1 + x)}{\overline{T}_s(t_1)} \geq \frac{\overline{T}_s(t_2 + x)}{\overline{T}_s(t_2)} ,\ 0 < t_1 \leq t_2 ,\ x \geq 0$$

$$\Leftrightarrow \frac{\overline{T}_s(t_1 + x)}{\overline{T}_s(t_2 + x)} \geq \frac{\overline{T}_s(t_1)}{\overline{T}_s(t_2)} ,\ 0 < t_1 \leq t_2 ,\ x \geq 0$$

$$\Leftrightarrow \frac{\overline{T}_s(x)}{\overline{T}_s(t + x)} \text{ is nondecreasing in } x \geq 0 \text{ , for each } t > 0$$

$$\Leftrightarrow X \overset{s-FR}{\geq} X_t ,\ t > 0$$

$$X \overset{s-FR}{\geq} X_t ,\ t > 0 \Leftrightarrow \frac{-\frac{d}{dx}\overline{T}_s(x+t)}{\overline{T}_s(x+t)} \leq \frac{-\frac{d}{dx}\overline{U}_s(x)}{\overline{U}_s(x)} ,\ x \geq 0$$

$$\Leftrightarrow r_{T_s}(x) \leq r_{U_s}(x) ,\ x \geq 0$$

$$\Leftrightarrow r_{T_s}(x) \leq r_{T_s}(t + x) \text{ , each } t,\ x \geq 0$$

$$\Leftrightarrow r_{T_s}(x + t_1) \leq r_{T_s}(x + t_2) ,\ 0 < t_1 \leq t_2,\ x \geq 0$$

$$\Leftrightarrow X_{t_1} \overset{s-FR}{\geq} X_{t_2} ,\ 0 < t_1 \leq t_2.$$

$$X_{t_1} \overset{s-FR}{\geq} X_{t_2} ,\ 0 < t_1 \leq t_2 \Leftrightarrow r_{T_s}(x + t_1) \leq r_{T_s}(x + t_2) ,\ 0 < t_1 \leq t_2,\ x \geq 0$$

$$\Leftrightarrow r_{T_s}(t_1) \le r_{T_s}(t_2)\ ,\ 0 < t_1 \le t_2$$

$$\Leftrightarrow r_{V_s}(0) \le r_{W_s}(0)$$

$$\Leftrightarrow X_{t_1} \overset{s-SFR}{\ge} X_{t_2}\ ,\ 0 < t_1 \le t_2 .$$

In the following corollary, we summarize the results regarding YBO and NBU type comparsions in terms of the various orders, and survey the relevance of Theorem 2.3 and classical orderings(see Deschpande et al(1986) and Choi et al(1995)).

Corollary 2.4:

1) X is 0-IFR (ILR) if and only if $X \overset{0-FR}{\ge} X_t$ $(X \overset{LR}{\ge} X_t)$, $t > 0$ if and only if $X_{t_1} \overset{0-FR}{\ge} X_{t_2}$ $(X_{t_1} \overset{LR}{\ge} X_{t_2})$ holds for all $0 < t_1 \le t_2$ if and only if $X_{t_1} \overset{0-ST}{\ge} X_{t_2}$ $(X_{t_1} \overset{WLR}{\ge} X_{t_2})$ holds for all $0 < t_1 \le t_2$.

2) X is 1-IFR (IFR) if and only if $X \overset{1-FR}{\ge} X_t$ $(X \overset{FR}{\ge} X_t)$, $t > 0$ if and only if $X_{t_1} \overset{1-FR}{\ge} X_{t_2}$ $(X_{t_1} \overset{FR}{\ge} X_{t_2})$ holds for all $0 < t_1 \le t_2$ if and only if $X_{t_1} \overset{1-ST}{\ge} X_{t_2}$ $(X_{t_1} \overset{ST}{\ge} X_{t_2})$ holds for all $0 < t_1 \le t_2$ if and only if $X_{t_1} \overset{1-SFR}{\ge} X_{t_2}$ $(X_{t_1} \overset{r(0)}{\ge} X_{t_2})$ holds for all $0 < t_1 \le t_2$.

3) X is 2-IFR (DMRL) if and only if $X \overset{2-FR}{\ge} X_t$ $(X \overset{MR}{\ge} X_t)$, $t > 0$ if and only if $X_{t_1} \overset{2-FR}{\ge} X_{t_2}$ $(X_{t_1} \overset{MR}{\ge} X_{t_2})$ holds for all $0 < t_1 \le t_2$ if and only if $X_{t_1} \overset{2-ST}{\ge} X_{t_2}$ $(X_{t_1} \overset{HAMR}{\ge} X_{t_2})$ holds for all $0 < t_1 \le t_2$ if and only if $X_{t_1} \overset{2-SFR}{\ge} X_{t_2}$ $(X_{t_1} \overset{E}{\ge} X_{t_2})$ holds for all $0 < t_1 \le t_2$.

Next, we consider that the YBO and NBU comparsions in terms of s-ST ordering and s-SFR ordering.

Theorem 2.5: If $X_{t_1} \overset{s-ST}{\ge} X_{t_2}$, $0 < t_1 \le t_2$, then $X \overset{s-ST}{\ge} X_t$, $t > 0$.

Proof. Using Theorem 2.3 and the fact that $X \overset{s-FR}{\ge} Y$ implies $X \overset{s-ST}{\ge} Y$.

Theorem 2.6: If $X_{t_1} \overset{s-SFR}{\geq} X_{t_2}$, $0 < t_1 \leq t_2$, then $X \overset{s-SFR}{\geq} X_t$, t > 0 .

Proof. Using Theorem 2.3 and the fact that $X \overset{s-FR}{\geq} Y$ implies $X \overset{s-SFR}{\geq} Y$.

In the following corollary, we survey connectios of Theorem 2.5, Theorem 2.6 and classical orderings(see Deschpande et al(1986)).

Corollary 2.7:

1) If $X_{t_1} \overset{1-ST}{\geq} X_{t_2}(X_{t_1} \overset{ST}{\geq} X_{t_2})$, $0 < t_1 \leq t_2$, then $X \overset{1-ST}{\geq} X_t (X \overset{ST}{\geq} X_t)$, t > 0 .

2) If $X_{t_1} \overset{1-SFR}{\geq} X_{t_2}(X_{t_1} \overset{r(0)}{\geq} X_{t_2}$), $0 < t_1 \leq t_2$, then $X \overset{1-SFR}{\geq} X_t$ ($X \overset{r(0)}{\geq} X_t$), t > 0 .

3) If $X_{t_1} \overset{2-SFR}{\geq} X_{t_2}(X_{t_1} \overset{E}{\geq} X_{t_2}$), $0 < t_1 \leq t_2$, then $X \overset{2-SFR}{\geq} X_t$ ($X \overset{E}{\geq} X_t$), t > 0 .

3. EQUILIBRIUM DISTRIBUTIONS

We consider a renewal process as generated by putting as item, with life distribution F, in use and the replacing it upon failure, the limiting distribution of residual life has distribution fuction $H_F(x) = \frac{1}{\mu_F} \int_0^x \overline{F}(u)du$,and density function $h_F(x) = \frac{\overline{F}(x)}{\mu_F}$. Let Z denote the random variable density $h_F(x)$ and $Z_t = (Z - t)|Z > t$ for any $t \geq 0$.

Using the fact the equilibrium distributions of order (s+1) of X is the equilibrium distribution of s-order equilibrium distribution, we have the following theorem.

Theorem 3.1:

1) $X \overset{(s+1)-FR}{\geq} X_t$, t > 0 if and only if $Z \overset{s-FR}{\geq} Z_t$, t > 0 .

2) $X \overset{(s+1)-ST}{\geq} X_t$, t > 0 if and only if $Z \overset{s-ST}{\geq} Z_t$, t > 0

In Reference Singh(1989) and Choi et al(1995), various positive ageing properties of X have been obtained in terms of positive ageing properties of Z. The following corollary gives the relevance of Theorem 3.1 and classical orderings.

Corollary 3.2: For all $t > 0$,

1) $X \overset{1-FR}{\geq} X_t$ $(X \overset{FR}{\geq} X_t)$, t > 0 if and only if $Z \overset{0-FR}{\geq} Z_t$ $(Z \overset{LR}{\geq} Z_t)$, t > 0 .

2) $X \overset{2-FR}{\geq} X_t$ $(X \overset{MR}{\geq} X_t)$, t > 0 if and only if $Z \overset{1-FR}{\geq} Z_t$ $(Z \overset{FR}{\geq} Z_t)$, t > 0 .

3) $X \overset{3-FR}{\geq} X_t$ $(X \overset{VR}{\geq} X_t)$, t > 0 if and only if $Z \overset{2-FR}{\geq} Z_t$ $(Z \overset{MR}{\geq} Z_t)$, t > 0 .

4) $X \overset{1-ST}{\geq} X_t$ $(X \overset{ST}{\geq} X_t)$, t > 0 if and only if $Z \overset{0-ST}{\geq} Z_t$ $(Z \overset{WLR}{\geq} Z_t)$, t > 0 .

5) $X \overset{2-ST}{\geq} X_t$ $(X \overset{HAMR}{\geq} X_t)$, t > 0 if and only if $Z \overset{1-ST}{\geq} Z_t$ $(Z \overset{ST}{\geq} Z_t)$, t > 0 .

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