Journal of Statistical
Theory & Methods
1996, Vol. 7, pp 93~104

Comparison of Interval Estimations for P(X<Y) in Marshall-Olkin's Model

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Abstracts In this paper, Marshall and Olkin's bivariate exponential distribution is assumed for stress and strength model. We derive the asymptotic distributions and construct some approximate confidence intervals for $P(X<Y)$ by some methods. Also we compare confidence intervals through Monte Carlo simulation.

keyword :Bootstrap method, Marshall and Olkin's model, Confidence interval.

1. Introduction

Let's consider the stress-strength model in the two component system. In this case, it is realistic to assume some forms of dependence among the components of system. This dependence among the components arise from common environmental shocks and stress. For these cases, the assumption that the components of systems have underlying bivariate exponential distribution proposed by Marshall and Olkin (1967) may be reasonable. The random variables (X,Y) are said to follow Marshall and Olkin's bivariate exponential distribution with parameters λ_1, λ_2 and λ_3 if

$$\overline{F}(x,y) = P(X > x, Y > y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_3 \max(x,y)\},\ x,y \geq 0, \quad (1.1)$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$. It follows that

$$P = P(X < Y) = \iint_{x<y} dF(x,y) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}. \quad (1.2)$$

The bivariate exponential distribution occupies an important place among bivariate life distributions in that it has the bivariate loss of memory property and

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its marginals have the loss of memory property. Also this model is applicable as a failure model for the systems where there exists positive probability of simultaneous failure of exponential components.

Arnold(1968) derived the unbiased estimators for parameters and its asymptotic property. Bemis, Bain and Higgins(1972) derived the moment type extimators for parameters. Awad, Azzam and Hamdon(1981) derived the maximum likelihood estimator(MLE), moment type estimator and Mann-Whitney type estimator for $P(X<Y)$. Kim and Park(1990) obtained the Bayes estimators of parameters and Bayes estimator of $P(X<Y)$. But the distribution function of an estimator for $P(X<Y)$ and a confidence interval of $P(X<Y)$ are not studied until now.

The bootstrap provides often ideal method to solve these problems. Efron (1979) initially introduced the bootstrap method to assign the accuracy for an estimator. Efron(1981, 1982, 1987) and Hall(1988) proposed some methods that construct an approximate confidence interval.

In this paper, we construct the distribution functions of MLE for $P(X<Y)$ and some approximate confidence intervals of $P(X<Y)$ based on the asymptotic normal and the bootstrap method. Also we compare the coverage probability and interval length of approximate confidence intervals through Monte Carlo simulation.

2. Preliminaries and Notations

We introduce the following notations for convenience.

BVED$(\lambda_1,\lambda_2,\lambda_3)$ = Marshall and Olkin's bivariate exponential distribution with parameters λ_1,λ_2 and λ_3.

$\underset{\sim}{x}=(x_1,x_2,\cdots,x_n)$: observation samples for stress.

$\underset{\sim}{y}=(y_1,y_2,\cdots,y_n)$: observation samples for strength.

$n_1=\sum_{i=1}^{n} I(x_i<y_i)$.

$n_2=\sum_{i=1}^{n} I(x_i>y_i)$.

$n_3=\sum_{i=1}^{n} I(x_i=y_i)$.

$\hat{\lambda}_i$ = MLE of λ_i, $i=1,2,3$.

$P=P(X<Y)$.

$\hat{P}$ = MLE of $P(X<Y)$.

$\underset{\sim}{x}^*=(x_1,x_2,\cdots,x_n)$: bootstrap samples for stress.

$\underline{y}^* = (y_1, y_2, \cdots, y_n)$: bootstrap samples for strength.

$\hat{\theta}^*$ =bootstrap estimator of θ

$\Phi(\cdot)$ =Cdf of the standard normal distribution.

$z^{(1-\alpha)}$ =the value such that $\Phi(x) = 1-\alpha$.

Consider a system with two components where lifetimes are X and Y from (1.1). By Bemis, Bain, and Higgins(1972), the likelihood function is expressed as follows:

$$L(\underline{x}, \underline{y}) = \exp\{-\lambda_1 \sum_{i=1}^{n} x_i - \lambda_2 \sum_{i=1}^{n} y_i - \lambda_3 \sum_{i=1}^{n} \max(x_i, y_i) + n_1 \ln[\lambda_1(\lambda_2 + \lambda_3)]$$
$$+ n_2 \ln[\lambda_2(\lambda_1 + \lambda_3)] + n_3 \ln \lambda_3\}. \quad (2.1)$$

Also, they have obtained the maximum likelihood equations as follows.

$$\frac{\partial}{\partial \lambda_1} \log L(\underline{x}, \underline{y}) = \frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_1 + \lambda_3} - \sum_{i=1}^{n} x_i = 0, \quad (2.2)$$

$$\frac{\partial}{\partial \lambda_2} \log L(\underline{x}, \underline{y}) = \frac{n_2}{\lambda_2} + \frac{n_1}{\lambda_2 + \lambda_3} - \sum_{i=1}^{n} y_i = 0, \quad (2.3)$$

$$\frac{\partial}{\partial \lambda_3} \log L(\underline{x}, \underline{y}) = \frac{n_1}{\lambda_2 + \lambda_3} + \frac{n_2}{\lambda_1 + \lambda_3} + \frac{n_3}{\lambda_3} - \sum_{i=1}^{n} \max(x_i, y_i) = 0. \quad (2.4)$$

By Proschan and Sullo(1976), the likelihood equations can be solved numerically using an iterative procedure. Awad, Azam and Hamdan(1981) derived three different estimators of p based on MLE, moment type estimator(MME) and Mann-Whitney type estimator(MWE), respectively. That is, MLE, MME and MWE of p are given by $\hat{\lambda}_1/(\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3)$, $(\overline{Y} - n_3\overline{X}/n)/(\overline{X} + \overline{Y})$ and n_1/n, respectively. Kim and Park(1990) derived Bayes estimators of p under various prior distributions. But, a distribution function of $\hat{p}$ and a confidence interval of p are not studied until now.

3. The Asymptotic Distributions

In this section, we derive the asymptotic distributions of the MLE of p.
By Hanagal and Kale(1991), $\sqrt{n}(\underline{\lambda} - \hat{\underline{\lambda}})$ has asymptotic trivariate normal distribution with mean vector zero and covariance matrix $\Sigma = (I^{ij}), i,j = 1,2,3$, where $(I^{ij}), i,j = 1,2,3$ is the (i,j)th element of inverse matrix of Fisher information $(I = (I_{ij}), i,j = 1,2,3)$ with

$$I_{11} = \frac{E(n_1)}{\lambda_1^2} + \frac{E(n_2)}{(\lambda_1+\lambda_3)^2},\ I_{12} = 0, I_{13} = \frac{E(n_2)}{(\lambda_1+\lambda_3)^2}$$
$$I_{22} = \frac{E(n_2)}{\lambda_2^2} + \frac{E(n_1)}{(\lambda_2+\lambda_3)^2},\ I_{23} = \frac{E(n_1)}{(\lambda_2+\lambda_3)^2}$$
$$I_{33} = \frac{E(n_1)}{\lambda_3^2} + \frac{E(n_1)}{(\lambda_2+\lambda_3)^2} + \frac{E(n_2)}{(\lambda_1+\lambda_3)^2}.$$

Since $\hat{\underline{\lambda}}$ has asymptotic trivariate normal distribution with mean $\underline{\lambda}$ and covariance matrix Σ / n, we can see that the asymptotic distribution of $\hat{P} = \hat{\lambda}_1/(\hat{\lambda}_1+\hat{\lambda}_2+\hat{\lambda}_3)$ is normal distribution with mean P and variance $D\Sigma D^T/n$ using the delta method(See Serfling(1980)). That is,

$$\sqrt{n}(\hat{P}-P) \xrightarrow{d} AN(0,\ D\Sigma D^T),$$

where $D = \partial \hat{P}/\partial \hat{\lambda}\,|_{\hat{\underline{\lambda}}=\underline{\lambda}} = \left((\lambda-\lambda_1)/\lambda^2,\ -\lambda_1/\lambda^2,\ -\lambda_1/\lambda^2\right)$ and $\lambda = \lambda_1+\lambda_2+\lambda_3$.
As alternative method, an approximate distribution of $\hat{P}$ can be constructed by the bootstrap method. The bootstrap procedure is as follows:
(1) Compute the MLE's of $\lambda_1, \lambda_2, \lambda_3$ by solving equations (2.2), (2.3) and (2.4), say $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$.
(2) Construct the sampling distribution from $\underline{x}$ and $\underline{y}$ based on $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$. That is, construct BVED$(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$.
(3) Generate bootstrap sample of size m from fixed BVED$(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$, say $(x_1^*, y_1^*), (x_2^*, y_2^*), \cdots, (x_m^*, y_m^*)$.
(4) Construct the likelihood equations same as (2.2), (2.3) and (2.4) based on bootstrap samples, that is,

$$\frac{n_1^*}{\hat{\lambda}_1} + \frac{n_2^*}{\hat{\lambda}_1+\hat{\lambda}_3} - \sum_{i=1}^{m} x_i^* = 0, \tag{3.1}$$

$$\frac{n_2^*}{\hat{\lambda}_2} + \frac{n_1^*}{\hat{\lambda}_2+\hat{\lambda}_3} - \sum_{i=1}^{m} y_i^* = 0, \tag{3.2}$$

$$\frac{n_1^*}{\hat{\lambda}_2+\hat{\lambda}_3} + \frac{n_2^*}{\hat{\lambda}_1+\hat{\lambda}_3} + \frac{n_3^*}{\hat{\lambda}_3} - \sum_{i=1}^{m} \max(x_i^*, y_i^*) = 0. \tag{3.3}$$

Where $n_i^*, i = 1,2,3$ is the bootstrap version of n_i. And compute the bootstrap estimators of $\lambda_1, \lambda_2, \lambda_3$ by solving equations (3.1), (3.2) and (3.3), say $\hat{\lambda}_1^*, \hat{\lambda}_2^*, \hat{\lambda}_3^*$.
(5) Compute the bootstrap estimator of P given as

$$\hat{P}^* = \frac{\hat{\lambda}_1^*}{\hat{\lambda}_1^* + \hat{\lambda}_2^* + \hat{\lambda}_3^*}. \tag{3.4}$$

(6) Repeat the step (3), (4) and (5), say repeat B times. We denote b th bootstrap estimator of P as $\hat{P}^{*b}, b = 1,2,\cdots,B$.

Theorem Let $\hat{P}^*$ and $\hat{\lambda}_i^*, i = 1,2,3$ be the bootstrap estimators of P and λ_i constructed by the bootstrap procedure. Then as $n,m \to \infty$, the asymptotic distribution of $\hat{P}^*$ and of $\hat{P}$ have the same.

Proof For arbitrary positive ε and $i = 1,2,3$,

$$P(|\hat{\lambda}_i^* - \lambda_i| \geq \varepsilon) \leq P(|\hat{\lambda}_i^* - \hat{\lambda}_i| \geq \varepsilon / 2) + P(|\hat{\lambda}_i - \lambda_i| \geq \varepsilon / 2). \tag{3.5}$$

The second term on the right side of (3.5) can be made arbitrary small as $n \to \infty$ since $\hat{\lambda}_i$ is the MLE of λ_i. The first term on the left side of (3.5) also can be made arbitrary small as $m \to \infty$ since $\hat{\lambda}_i^*$ is the MLE's of $\hat{\lambda}_i$ by the bootstrap procedure. Therfore, $\hat{\lambda}_i^*$ is a consistent estimator of λ_i. Hence, $\hat{P}^*$ is also a consistent estimator of P. Hence, the proof is completed.

we can obtain an approximate bootstrap distribution function of $\hat{P}$, say $\hat{F}^*(s)$ as follows:

$$\hat{F}^*(s) = \frac{1}{B} \sum_{b=1}^{B} I(\hat{P}^{*b} \leq s), \tag{3.6}$$

where s is arbitrary real number.

4. Interval Estimations

In this section, we construct approximate confidence intervals of P based on the aymptotic normal distribution and the bootstrap procedure. All confidence intervals for P are two-sided and equal-tailed with confidence level $100(1-2\alpha)\%$.

4.1 Normal method

Since the asymptotic distribution of $\sqrt{n}(\hat{P} - P)$ is normal distribution with mean zero and variance $D\Sigma D^T$, we can construct $100(1-2\alpha)\%$ approximate confidence interval(normal interval) based on normal distribution for P as follows:

$$(\hat{P} + z^{(\alpha)} \cdot \sqrt{\hat{D}\hat{\Sigma}\hat{D}^T / n}, \ \hat{P} + z^{(1-\alpha)} \cdot \sqrt{\hat{D}\hat{\Sigma}\hat{D}^T / n}), \tag{4.1}$$

where $z^{(\alpha)}$ is 100α percentile of standard normal distribution.

Hence, the coverage probability(CP_{nor}) and interval length(IL_{nor}) for percentile interval can be computed as follows:

$$CP_{nor} = P(\hat{P} + z^{(\alpha)} \cdot \sqrt{\hat{D}\hat{\Sigma}\hat{D}^T / n} < P < \hat{P} + z^{(1-\alpha)} \cdot \sqrt{\hat{D}\hat{\Sigma}\hat{D}^T / n}) \tag{4.2}$$

and

$$IL_{nor} = (\hat{P} + z^{(1-\alpha)} \cdot \sqrt{\hat{D}\hat{\Sigma}\hat{D}^T / n}) - (\hat{P} + z^{(\alpha)} \cdot \sqrt{\hat{D}\hat{\Sigma}\hat{D}^T / n}) \tag{4.3}$$

4.2. The Bootstrap method

Percentile method

The percentile interval by Efron(1981) is obtained by percentiles of the empirical bootstrap distribution function of $\hat{P}^*$. Let $\hat{F}^{*-1}(\alpha)$ be the 100α empirical percentile of $\hat{P}^*$ given as

$$\hat{F}^{*-1}(\alpha) = \inf\{s : \hat{F}^*(s) \geq \alpha\}. \tag{4.4}$$

That is, $\hat{F}^{*-1}(\alpha)$ is the $B\alpha$th value in the odered list of the B replications of $\hat{P}^{*b}$. If $B\alpha$ is not an integer, we can take the largest integer that less than or equal to $(B+1)\alpha$. Then $100(1-2\alpha)\%$ percentile interval for P is approximated by

$$\left(\hat{F}^{*-1}(\alpha)\ ,\ \hat{F}^{*-1}(1-\alpha)\right). \tag{4.5}$$

Hence, the coverage probability(CP_{per}) and interval length(IL_{per}) for percentile interval can be computed as follows:

$$CP_{per} = P\{\hat{F}^{*-1}(\alpha) < P < \hat{F}^{*-1}(1-\alpha)\} \tag{4.6}$$

and

$$IL_{per} = \hat{F}^{*-1}(1-\alpha) - \hat{F}^{*-1}(\alpha). \tag{4.7}$$

Bias correct method

The bias correct interval(BC interval) by Efron(1982) adjusts a possible bias in estimating P. The bias correction is estimated as

$$\hat{z}_0 = \Phi^{-1}(\hat{F}^*(\hat{P}^*)) = \Phi^{-1}\left(\frac{1}{B}\sum_{b=1}^{B} I(\hat{P}^{*b} \leq \hat{P})\right) \tag{4.8}$$

Therefore, we have $100(1-2\alpha)\%$ approximate BC interval for P given as

$$\left(\hat{F}^{*-1}(\Phi(2\hat{z}_0 + z^{(\alpha)}))\ ,\ \hat{F}^{*-1}(\Phi(2\hat{z}_0 + z^{(1-\alpha)}))\right). \tag{4.9}$$

Hence, the coverage probability(CP_{BC}) and interval length(IL_{BC}) for BC interval can be computed as follows:

$$CP_{BC} = P\left(\hat{F}^{*-1}(\Phi(2\hat{z}_0 + z^{(\alpha)})) < P < \hat{F}^{*-1}(\Phi(2\hat{z}_0 + z^{(1-\alpha)}))\right). \tag{4.10}$$

and

$$IL_{BC} = \hat{F}^{*-1}(\Phi(2\hat{z}_0 + z^{(1-\alpha)})) - \hat{F}^{*-1}(\Phi(2\hat{z}_0 + z^{(\alpha)})). \tag{4.11}$$

Bias correct acceleration method

The bias correct acceleration interval(BCa interval) by Efron(1987) corrects both the bias and standard error for $\hat{p}$. That is, the BCa interval requires to calculate the bias-correction constant $\hat{z}_0$ and the acceleration constant $\hat{a}$. In fact, the bias-correction constant $\hat{z}_0$ is the same as that of BC method. There are various ways to compute the acceleration constant $\hat{a}$. We will use the acceleration constant $\hat{a}$ given in term of the jackknife values of $\hat{p}$. Let $\hat{P}_{(i)} = \hat{\lambda}_{1(i)} / (\hat{\lambda}_{1(i)} + \hat{\lambda}_{2(i)} + \hat{\lambda}_{3(i)})$ and define $\hat{P}_{(\cdot)} = \sum_{i=1}^{n} \hat{P}_{(i)}/n$, where $\hat{\lambda}_{j(i)}$, $j = 1,2,3$ denote $\hat{\lambda}_{j(i)}$ computed from the original sample with the i th point (x_i, y_i) deleted. Then we can compute the acceleration constant $\hat{a}$ given as

$$\hat{a} = \sum_{i=1}^{n} (\hat{P}_{(\cdot)} - \hat{P}_{(i)})^3 / 6\{\sum_{i=1}^{n} (\hat{P}_{(\cdot)} - \hat{P}_{(i)})^2\}^{3/2}. \tag{4.12}$$

Therefore, we have $100(1-2\alpha)\%$ approximate BCa interval for p by

$$\left(\hat{F}^{*-1}(\alpha_1)\ ,\ \hat{F}^{*-1}(\alpha_2)\right), \tag{4.13}$$

where $\alpha_1 = \Phi\left(\hat{z}_0 + (\hat{z}_0 + z^{(\alpha)}) / \{1 - \hat{a}(\hat{z}_0 + z^{(\alpha)})\}\right)$ and

$\alpha_2 = \Phi\left(\hat{z}_0 + (\hat{z}_0 + z^{(1-\alpha)}) / \{1 - \hat{a}(\hat{z}_0 + z^{(1-\alpha)})\}\right)$.

Hence, the coverage probability(CP_{BCa}) and interval length(IL_{BCa}) for BCa interval can be computed as follows:

$$CP_{BCa} = P\{\hat{F}^{*-1}(\alpha_2) < P < \hat{F}^{*-1}(\alpha_1)\} \tag{4.14}$$

and

$$IL_{BCa} = \hat{F}^{*-1}(\alpha_2) - \hat{F}^{*-1}(\alpha_1). \tag{4.15}$$

Percentile-t method

The percentile-t interval by Hall(1988) is constructed by using the bootstrap distribution of an approximately pivotal quantity for $\hat{P}$. Since the limiting distribution of $\sqrt{n}(\hat{P}-P)/\sqrt{U\Sigma U^T}$ does not depend on unknown parameters, we can construct a bootstrap pivotal quantity for $\hat{P}$ by $\hat{P}^*_t=\sqrt{n}(\hat{P}^*-\hat{P})/\sqrt{\hat{U}^*\hat{\Sigma}^*\hat{U}^{*T}}$, where $\hat{U}^*$ and $\hat{\Sigma}^*$ are the values of $\hat{U}$ and $\hat{\Sigma}$ based on bootstrap samples, respectively. Let $\hat{F}_t^*$ denote the empirical distribution function of $\hat{P}_t^*$. Then we can compute $\hat{F}_t^*$ as follows:

$$\hat{F}_t^*(s)=\frac{1}{B}\sum_{b=1}^{B} I(\hat{P}_t^{*b}\le s). \tag{4.16}$$

Therefore, we have $100(1-2\alpha)\%$ approximate percentile-t interval for P as follows:

$$(\hat{P}+\hat{F}_t^{*-1}(\alpha)\cdot\sqrt{\hat{D}\hat{\Sigma}\hat{D}^T/n},\ \hat{P}+\hat{F}_t^{*-1}(1-\alpha)\cdot\sqrt{\hat{D}\hat{\Sigma}\hat{D}^T/n}).$$

(4.17)

Hence, the coverage probability(CP_{per-t}) and interval length(IL_{per-t}) for percentile-t interval can be computed as follows:

$$CP_{per-t}=P(\hat{P}+\hat{F}_t^{*-1}(\alpha)\cdot\sqrt{\hat{D}\hat{\Sigma}\hat{D}^T/n}<P<\hat{P}+\hat{F}_t^{*-1}(1-\alpha)\cdot\sqrt{\hat{D}\hat{\Sigma}\hat{D}^T/n}) \tag{4.18}$$

and

$$IL_{per-t}=(\hat{P}+\hat{F}_t^{*-1}(1-\alpha)\cdot\sqrt{\hat{D}\hat{\Sigma}\hat{D}^T/n}-(\hat{P}+\hat{F}_t^{*-1}(\alpha)\cdot\sqrt{\hat{D}\hat{\Sigma}\hat{D}^T/n}). \tag{4.19}$$

5. Comparisons and Conclusions

To compare the approximate bootstrap confidence interval estimates, we first will compute the results obtained in Section 3. The methods are compared based mainly on coverage probability and interval length. The Marshall and Olkin's bivariate exponential random numbers were generated by the method proposed by Friday and Patil(1977). For $\lambda_2=0.05$ and $\lambda_3=0.1$, the values of λ_1 are selected so that the values of P are 0.1, 0.3, 0.5, 0.7 and 0.9. That is, $\lambda_1=0.017, 0.065, 0.15, 0.35$ and 1.35. Sample sizes n are 5, 10, 20 and 40 and the used confidence level $(1-2\alpha)$ is 0.90 and 0.95. For given independent random samples the approximate confidence intervals were constructed by each method with bootstrap replications $B=1000$ times. And the Monte Carlo samplings were repeated 500 times.

We can note the following properties through Table and Figures.

Table 1. Coverage Probabilities for Proposed Intervals

n	$P(X<Y)$	2α	CP_{nor}	CP_{per}	CP_{BC}	CP_{BCa}	CP_{per-t}
10	0.1	0.10	0.6480	0.6600	0.6540	0.6520	0.6720
		0.05	0.6680	0.6740	0.6880	0.6780	0.6740
10	0.3	0.10	0.7860	0.8220	0.8200	0.7980	0.8000
		0.05	0.8260	0.9060	0.8600	0.8660	0.8200
10	0.5	0.10	0.8400	0.8680	0.8740	0.8800	0.8320
		0.05	0.9020	0.9120	0.9160	0.9300	0.8960
10	0.7	0.10	0.8740	0.9020	0.8920	0.9040	0.8740
		0.05	0.9260	0.9380	0.9400	0.9480	0.9060
10	0.9	0.10	0.8440	0.8720	0.8740	0.8680	0.8960
		0.05	0.8800	0.9080	0.9080	0.9140	0.9460
20	0.1	0.10	0.7680	0.7840	0.8640	0.8720	0.8460
		0.05	0.8260	0.8320	0.9010	0.9120	0.8990
20	0.3	0.10	0.8560	0.9020	0.9180	0.9120	0.8800
		0.05	0.9260	0.9420	0.9560	0.9540	0.9300
20	0.5	0.10	0.8660	0.8760	0.8800	0.8920	0.8520
		0.05	0.9100	0.9240	0.9380	0.9380	0.9040
20	0.7	0.10	0.8860	0.8960	0.9000	0.8900	0.8960
		0.05	0.9440	0.9580	0.9580	0.9580	0.9280
20	0.9	0.10	0.8640	0.9140	0.9080	0.9060	0.9260
		0.05	0.9140	0.9360	0.9340	0.9400	0.9680
40	0.1	0.10	0.8220	0.8280	0.8420	0.8680	0.8680
		0.05	0.8780	0.8780	0.9010	0.9220	0.9210
40	0.3	0.10	0.8800	0.8920	0.8960	0.8960	0.8980
		0.05	0.9340	0.9360	0.9520	0.9540	0.9540
40	0.5	0.10	0.8700	0.8720	0.8860	0.8880	0.8740
		0.05	0.9380	0.9400	0.9420	0.9520	0.9320
40	0.7	0.10	0.8940	0.8980	0.9060	0.9020	0.8980
		0.05	0.9420	0.9520	0.9580	0.9540	0.9420
40	0.9	0.10	0.8880	0.8980	0.8940	0.8980	0.9080
		0.05	0.9400	0.9380	0.9400	0.9420	0.9700

(1) The coverage probabilities of bootstrap intervals are slightly work better than that of normal interval.

(2) As a whole, the interval length by percentile-t method is longer than those of other intervals for small sample.

(3) For fixed sample size, the coverage probabilities of all approximate intervals

Table 2. Interval Lengths for Proposed Intervals

n	$P(X<Y)$	2α	IL_{nor}	IL_{per}	IL_{BC}	IL_{BCa}	IL_{per-t}
10	0.1	0.10	0.2416	0.2289	0.2227	0.2277	0.5521
		0.05	0.2880	0.2618	0.2540	0.2685	0.5740
10	0.3	0.10	0.3930	0.3787	0.3736	0.3786	0.5521
		0.05	0.4688	0.4407	0.4396	0.4491	0.6822
10	0.5	0.10	0.4085	0.4066	0.4049	0.4064	0.6873
		0.05	0.4872	0.4781	0.4802	0.4834	0.7902
10	0.7	0.10	0.3271	0.3340	0.3310	0.3284	0.3652
		0.05	0.3901	0.3999	0.3991	0.3962	0.4541
10	0.9	0.10	0.1413	0.1758	0.1595	0.1578	0.1596
		0.05	0.1685	0.2288	0.2023	0.1999	0.2046
20	0.1	0.10	0.2033	0.1936	0.1896	0.1905	0.5190
		0.05	0.2424	0.2259	0.2238	0.2232	0.5580
20	0.3	0.10	0.3068	0.3037	0.3062	0.3092	0.3863
		0.05	0.3659	0.3585	0.3631	0.3677	0.6189
20	0.5	0.10	0.3054	0.3045	0.3064	0.3068	0.3279
		0.05	0.3642	0.3637	0.3657	0.3660	0.4052
20	0.7	0.10	0.2382	0.2408	0.2420	0.2413	0.2511
		0.05	0.2841	0.2898	0.2913	0.2893	0.3073
20	0.9	0.10	0.0984	0.1175	0.1102	0.1097	0.1093
		0.05	0.1174	0.1512	0.1412	0.1401	0.1383
40	0.1	0.10	0.1533	0.1498	0.1518	0.1525	0.2957
		0.05	0.1828	0.1759	0.1796	0.1799	0.3676
40	0.3	0.10	0.2200	0.2186	0.2204	0.2218	0.2327
		0.05	0.2624	0.2606	0.2633	0.2649	0.2856
40	0.5	0.10	0.2164	0.2185	0.2202	0.2201	0.2262
		0.05	0.2617	0.2615	0.2637	0.2630	0.2745
40	0.7	0.10	0.1658	0.1662	0.1674	0.1668	0.1694
		0.05	0.1977	0.1997	0.2007	0.2002	0.2046
40	0.9	0.10	0.0698	0.0737	0.0734	0.0732	0.0725
		0.05	0.0832	0.0935	0.0922	0.0918	0.0902

don't work well for small value of p.

(4) In most cases, all coverage probabilities of approximate confidence intervals tend to converge to true confidence level $(1-2\alpha)$. Also, all interval lengths of approximate confidence intervals tend to converge to true interval length.

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