JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 9, July 1996

POISSON STRUCTURE OF THE QUANTUM AFFINE SPACE

KANGJU MIN AND SEI-QWON OH

The aim of this note is to construct a Poisson algebra A such that there is a bijection between the set of symplectic ideals of A and the set of primitive ideals of the coordinate ring of quantum affine space, denoted by $\mathcal{O}_q(k^n)$, in the case when $q \in k^*$ is not a root of unity and k is an uncountably infinite and algebraically closed field with characteristic zero. The primitive ideals of $\mathcal{O}_q(k^n)$ were classified in [4] and S.P. Smith suggested that the primitive ideals of certain algebras related to quantum groups should correspond bijectively to the symplectic leaves of a naturally associated Poisson structure on the associated algebraic variety. Hence this note confirms S.P. Smith's suggestion for $\mathcal{O}_q(k^n)$.

The coordinate ring $\mathcal{O}_q(k^n)$ of quantum affine space is the algebra over a field k generated by $x_1, ..., x_n$ subject to the relations $x_jx_i = qx_ix_j$, $i < j$, for some $q \in k^*$. The quantum matrices algebras and the quantum enveloping algebra of a semisimple Lie algebra act on $\mathcal{O}_q(k^n)$. The reader is reffered to [4] or [5] for the further background and these actions on the algebra $\mathcal{O}_q(k^n)$. Henceforth we shall assume that $q \in k^*$ *is not a root of unity* and the ground field k *is an uncountably infinite and algebraically closed with characteristic zero.*

Let P be a free abelian group with finite rank n and let $\sigma : P \times$

The present study was supported by BSRI-95-1427, Ministry of Education.
Received by the editors on June 30, 1996.
1991 *Mathematics subject classifications*: Primary 16D60.

$P \longrightarrow k^*$ be an antisymmetric bimultiplicative map. That is,

$$\sigma(\lambda_1 + \lambda_2, \mu) = \sigma(\lambda_1, \mu)\sigma(\lambda_2, \mu)$$
$$\sigma(\lambda, \mu) = \sigma(\mu, \lambda)^{-1}.$$

Then σ is a 2-cocycle and thus we have the twisted group algebra $k^\sigma P$ that is defined by the commutation relations:

$$t_\lambda t_\mu = \sigma^2(\lambda, \mu) t_\mu t_\lambda$$

Define

$$P_\sigma = \{\lambda \in P \mid \sigma^2(\lambda, \mu) = 1 \quad \forall \mu \in P\}.$$

Clearly P_σ is a free abelian subgroup of P.

LEMMA 1. *The centre $Z(k^\sigma P)$ of $k^\sigma P$ is*

$$Z(k^\sigma P) = \{\sum_\lambda a_\lambda t_\lambda \mid \lambda \in P_\sigma\},$$

which is isomorphic to the group algebra kP_σ.

PROOF. Put $Z = Z(k^\sigma P)$. For $f = \sum_\lambda a_\lambda t_\lambda \in k^\sigma P$, $f \in Z$ if and only if $t_\mu f = f t_\mu$ for all $\mu \in P$. Since $t_\mu f = \sum_\lambda \sigma^2(\mu, \lambda) a_\lambda t_\lambda t_\mu$, this will occur if and only if $\lambda \in P_\sigma$ for all λ in the support of f.

THEOREM 2. *There is a bijection preserving inclusions between the ideals of $k^\sigma P$ and the ideals of the center $Z(k^\sigma P)$. That is, if I is an ideal of $k^\sigma P$ then $I = (I \cap Z(k^\sigma P))k^\sigma P$, and if J is an ideal of $Z(k^\sigma P)$ then $J = Jk^\sigma P \cap Z(k^\sigma P)$.*

PROOF. Consider the action of P as automorphisms of $k^\sigma P$ defined by

$$\lambda(t_\mu) = \sigma^2(\lambda, \mu) t_\mu = t_\lambda t_\mu t_\lambda^{-1}.$$

Let $\mathcal{T}$ be a transversal for P_σ in P. Then the weight space decomposition of $k^\sigma P$ under this action is

$$(*) \qquad k^\sigma P = \bigoplus_{\nu \in \mathcal{T}} Z(k^\sigma P)t_\nu$$

If I is an ideal of $k^\sigma P$ then I must be invariant under this action and so

$$I = \bigoplus_\nu I \cap Z(k^\sigma P)t_\nu = \bigoplus_\nu (I \cap Z(k^\sigma P))t_\nu = (I \cap Z(k^\sigma P))k^\sigma P.$$

If J is an ideal of $Z(k^\sigma P)$ and $x \in Jk^\sigma P \cap Z(k^\sigma P)$ then $x = \sum_i x_i f_i$ for some $x_i \in J$ and $f_i \in k^\sigma P$. Replace each f_i to an element written by the decomposition (*) and then x can be expressed by $x = \sum_{\nu \in \mathcal{T}} a_\nu t_\nu$ for some $a_\nu \in J$. Since $x \in Z(k^\sigma P)$, if $\nu \notin P_\sigma$ then $a_\nu = 0$ and so $x \in J$. Therefore we have that $J = Jk^\sigma P \cap Z(k^\sigma P)$.

Note that $\mathcal{O}_q(k^n)$ can be expressed as an n-fold iterated skew polynomial ring starting with the field k; hence, $\mathcal{O}_q(k^n)$ is an affine domain. We write $P_q(k^n)$ for the localization of $\mathcal{O}_q(k^n)$ with respect to the multiplicative set generated by $x_1, \ldots, x_n$.

LEMMA 3. *For a free abelian group P with basis $\{e_1, \ldots, e_n\}$, let σ be an antisymmetric bimultiplicative map on P defined by*

$$\sigma(e_i, e_j) = \begin{cases} q^{-1/2} & i < j \\ 1 & i = j \\ q^{1/2} & i > j \end{cases}$$

Then $P_q(k^n) \cong k^\sigma P$ and the center $Z(k^\sigma P) \cong kP_\sigma$, where

$$P_\sigma = \begin{cases} \{0\} & \text{if } n \text{ is even} \\ \mathbb{Z}(e_1 - e_2 + \cdots - e_{n-1} + e_n) & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. It is easy to check that the map from $P_q(k^n)$ into $k^\sigma P$ defined by $x_i \mapsto t_{e_i}$, $i = 1, \ldots, n$ is an isomorphism and the fact $Z(k^\sigma P) \cong kP_\sigma$ follows from Lemma 1. Moreover, P_σ is found easily by a straight calculation.

COROLLARY 4. *If n is even then $\langle 0 \rangle$ is unique minimal primitive ideal of $\mathcal{O}_q(k^n)$ and any other primitive ideal contains some x_i. If n is odd then all ideals of the form*

$$\langle x_1 x_3 \cdots x_n - \alpha x_2 x_4 \cdots x_{n-1} \rangle, \ \alpha \in k^*$$

are all minimal primitive ideals of $\mathcal{O}_q(k^n)$ and any other primitive ideal contains some x_i.

PROOF. If n is even then, by Lemma 3, every nonzero prime ideal of $\mathcal{O}_q(k^n)$ contains some x_i since x_i are normal and so $\langle 0 \rangle$ is the unique minimal primitive ideal by [**2**, 9.1.8(i)].

Let n be odd. Then all maximal ideals of $Z(k^\sigma P)$, where σ and P are ones of Lemma 3, are of the form

$$\langle t_{e_1} t_{e_2}^{-1} \cdots t_{e_{n-1}}^{-1} t_{e_n} - \alpha \rangle, \ \alpha \in k^*$$

by Theorem 2. Therefore the conclusion follows since $P_q(k^n)$ is the localization of $\mathcal{O}_q(k^n)$ with respect to the multiplicative set generated by $x_1, \ldots, x_n$.

Let σ, P and $e_1, \ldots, e_n$ be as in Lemma 3. Since $q \in k^*$ is not a root of unity, the map u from $P \times P$ into k subject to

$$\sigma^2(\lambda, \mu) = q^{u(\lambda, \mu)} \ \ \forall \lambda, \mu \in P$$

is antisymmetric and bilinear. That is,

$$u(\lambda_1 + \lambda_2, \mu) = u(\lambda_1, \mu) + u(\lambda_2, \mu)$$
$$u(\lambda, \mu) = -u(\mu, \lambda)$$

More precisely, the antisymmetric and bilinear map u is defined by

$$u(e_i, e_j) = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases}$$

Since u is an antisymmetric and bilinear map, there is a Poisson structure on the group algebra kP define by

$$\{t_\lambda, t_\mu\} = u(\lambda, \mu) t_{\lambda+\mu}.$$

More precisely,

$$\{t_{e_i}, t_{e_j}\} = \begin{cases} -t_{e_i+e_j} & i < j \\ 0 & i = j \\ t_{e_i+e_j} & i > j. \end{cases}$$

Hence the subalgebra of kP generated by $t_{e_1}, ..., t_{e_n}$, denoted by $A(n)$, is a sub-Poisson algebra of kP.

LEMMA 5. *Set*

$$Z_p(kP) = \{f \in kP \mid \{f, g\} = 0 \ \forall g \in kP\}.$$

Then $Z_p(kP) = kP_u$ *where* $P_u = \{\lambda \in P \mid u(\lambda, \mu) = 0 \ \forall \mu \in P\}$. *The Poisson subalgebra* $Z_p(kP)$ *of* kP, *which has the trivial Poisson structure, is called the Poisson center.*

PROOF. Let $f = \sum_\lambda a_\lambda t_\lambda$. Then $f \in Z_p(kP)$ if and only if $\{t_\mu, f\} = 0$ for all $\mu \in P$. Since $\{t_\mu, f\} = \sum_\lambda a_\lambda u(\mu, \lambda) t_{\mu+\lambda}$, this will occur if and only if $\lambda \in P_u$ for all λ in the support of f.

Recall that a Poisson ideal of a Poisson algebra A is an ideal I such that $\{f, g\} \in I$ for all $f \in I$ and $g \in A$.

THEOREM 6. *There is a bijection preserving inclusions between the Poisson ideals of kP and the Poisson ideals of $Z_p(kP)$. That is, if I is a Poisson ideal of kP then $I = (I \cap Z_p(kP))kP$, and if J is a Poisson ideal of $Z_p(kP)$ then $J = (JkP) \cap Z_p(kP)$.*

PROOF. Set $Z_p = Z_p(kP)$. Consider the action of P as linear endomorphisms of kP defined by

$$\lambda(t_\mu) = u(\lambda, \mu)t_\mu = \{t_\lambda, t_\mu\}t_\lambda^{-1}.$$

Let T be a transversal for P_u in P. Then the weight space decomposition of kP under this action is

$$(**) \qquad kP = \bigoplus_{\nu \in T} Z_p t_\nu$$

If I is a Poisson ideal of kP then I must be invariant under this action and so

$$I = \bigoplus_{\nu} I \cap Z_p t_\nu = \bigoplus_{\nu} (I \cap Z_p)t_\nu = (I \cap Z_p)kP.$$

If J is a Poisson ideal of Z_p and if $x \in (JkP) \cap Z_p$ then $x = \sum_i x_i f_i$ for some $x_i \in J$ and $f_i \in kP$. Replace each f_i to an element written by the decomposition (**) and then x can be expressed by $x = \sum_{\nu \in T} a_\nu t_\nu$ for some $a_\nu \in J$. Since $x \in Z_p$, if $\nu \notin P_u$ then $a_\nu = 0$ and so $x \in J$. Therefore we have that $J = (JkP) \cap Z_p$.

Let A be a Poisson algebra over k and let Q be a prime Poisson ideal of A. Then the Poisson bracket on A defines a Poisson bracket on Fract(A/Q) and define Q to be symplectic if $\{a \in \text{Fract}(A/Q) \mid \{a, b\} = 0\ \forall b \in \text{Fract}(A/Q)\}$ reduces to scalars (see [**1**, A.4.1]). A Poisson algebra A is called symplectic whenever the Poisson ideal $\langle 0 \rangle$ is symplectic.

THEOREM 7. *There is a bijection between Prim($\mathcal{O}_q(k^n)$), the set of all primitive ideals, and Symp($A(n)$), the set of all symplectic ideals.*

PROOF. We proceed the proof by induction on n. If $n = 1$ then the conclusion is true because $\mathcal{O}_q(k^1) = k[x] \cong A(1)$ and the Poisson structure of $A(1)$ is trivial. Assume that $n > 1$ and Theorem is true for positive integers less than n.

Note that the center $Z(k^\sigma P)$ is isomorphic to the Poisson center of kP since $\sigma^2(\lambda, \mu) = q^{u(\lambda,\mu)}$ and q is not a root of unity. If n is even then $A(n)$ is symplectic since $\mathrm{Frac}(A(n)) = \mathrm{Frac}(kP)$ and every nonzero symplectic ideal of $A(n)$ contains some t_{e_i} by Theorem 6. If n is odd then any symplectic ideal of $A(n)$ which is not containing any t_{e_i} contains

$$f = t_{e_1} t_{e_3} \cdots t_{e_n} - \alpha t_{e_2} t_{e_4} \cdots t_{e_{n-1}}$$

for some $\alpha \in k^*$ by Theorem 6 and the ideal $\langle f \rangle$ is symplectic. Therefore the proof is complete by Corollary 4 and the induction n.

REFERENCES

1. A. Joseph, *Quantum Groups and Their Primitive Ideals*, 3. Folge-Band 29, A series of modern surveys in mathematics, Springer-Verlag, 1995.
2. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, New York, 1987.
3. Sei-Qwon Oh, *Primitive ideals of the coordinate ring of quantum symplectic space*, J. Algebra **174** (1995), 531–552.
4. S. P. Smith, *Quantum groups: An introduction and survey for ring theorists*, in Noncommutative Rings (S. Montgomery and L. Small, eds.), M.S.R.I. Publ. 24, Springer-Verlag, New York, 1992, pp. 131–178.
5. M. Takeuchi, *Matric bialgebras and quantum groups*, Israel J. of Math. **72** (1990), 232-251.

Department of Mathematics
Chungnam National University
Taejon 305-764, Korea