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THE CLOSING LEMMA FOR CHAIN RECURRENCE IN NONCOMPACT MANIFOLDS

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Solving the Closing Lemma is interesting in its own right, but more because it implies generically that a dynamical system already has its periodic orbits dense in its set of nonwandering orbits. The first statement and proof of C^1 Closing Lemma (for nonwandering points) are due to Pugh [6]. There was a gap in his proof which was repaired in [8]. As proved by Pugh and Robinson, the C^1 Closing Lemma states that if p is a nonwandering point of a C^1 vector field X on a compact manifold M then every neighborhood of X in the C^1 topology contains a vector field Y having a periodic orbit through p. The C^r Closing Lemma says that if X is C^r then Y can be found in any C^r neighborhood of X, $r \geq 0$.

For $r > 1$ the C^r Closing Lemma has not yet to be verified, even generically, and is known only for very special cases. For detailed historical comments, see [3,5,6,7,8].

Recently, there were some attempts to solve the Closing Lemma for more generealzed recurrence in special spaces (for example, in the plane $\mathbb{R}^2$ or in the noncompact surface) (see [3,5,7]).

In 1988, Peixoto showed that if X is a C^r vector field on the plane $\mathbb{R}^2$ and every fixed point of X is semihyperbolic or satisfy the shadowing property, then every prolongationally recurrent point p of X

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can be periodic under a C^r perturbation of X in the C^r Whitney topology (for more details, see [3]).

More recently, Peixoto and Pugh [5] improved the above results at the chain recurrent points, which are the weakest type of recurrence in dynamics theory. To be precise, they proved that any chain recurrent point of a C^r vector field X on the plane $\mathbb{R}^2$ can be periodic under a C^r perturbation of X in the C^r Whitney topology if every fixed point of X is hyperbolic.

The purpose of this paper is to study the Closing Lemma at chain recurrent points of flows on a noncompact manifold M. Consequently we show that the C^0 Closing Lemma for chain recurrence in a noncompact manifold M does hold, but the C^1 Closing Lemma for chain recurrence in a noncompact manifold does not hold.

In what follows, let M be a connected, finite-dimensional C^∞-manifold which is not compact. A *flow* on M is a continuous map $\phi : M \times \mathbb{R} \to M$ such that

(1) $\phi(p, 0) = p$,
(2) $\phi(\phi(p,t),s) = \phi(p, s+t),\ p \in M,\ s,t \in \mathbb{R}$.

It follows easily that the transition map $\phi_t : M \to M$ defined by $\phi_t(p) = \phi(p,t),\ t \in \mathbb{R}$, are homeomorphisms.

The *orbit* of $p \in M$ under ϕ is set

$$O(p) = \{\phi_t(p) : t \in \mathbb{R}\}$$

We say that a point $p \in M$ is *periodic* for ϕ if $p \in O(p) \neq \{p\}$. The set of all periodic points for ϕ is denoted by $\mathcal{P}(\phi)$.

There are several ways to extend the concept of periodicity. The simplest is based on the *ω-limit set* of p,

$$\omega(p) = \{q \in M : \phi_{t_n}(p) \to q \ forsome\ t_n \to \infty\}$$

A point $p \in M$ is called *recurrent* if $p \in \omega(p)$. Although $\omega(p)$ is closed and ϕ-invariant, there may be points q near it which tends to new limits. This leads one to define the *first positive prolongational limit set* of p,

$$J^+(p) = \{y \in M : \phi_{t_k}(x_k) \to y \text{ for some } x_k \to p,\ t_k \to \infty\}.$$

For any subset A of M, we let $J^+(A) = \cup_{p\in A} J^+(p)$. For each $n \geq 1$, we denote the set $J_n(p)$ as follows.

(1) $J_1(p) = J^+(p)$
(2) $J_n(p) = J^+(J_{n-1}(p)),\ n \geq 2$

A point $p \in M$ is said to be *nonwandering* if $p \in J^+(p)$, and is called *generalized recurrent* (or *prolongationally recurrent*) if $p \in J_n(p)$ for some $n \geq 1$. The set of all nonwandering points (or generalized recurrent points) for ϕ will be denoted by $\Omega(\phi)$ (or $R(\phi)$), respectively. It is easy to show that a point $p \in M$ is nonwandering if and only if for any neighborhood U of p in M and any $t \in \mathbb{R}$ there exists $T > t$ with $\phi_T(U) \cap U \neq \emptyset$.

Surprisingly, a far more weak type of periodicity exists based on the concepts of chains. Here we give the concept of chain recurrence in a flow ϕ on a noncompact manifold M, which is introduced by Peixoto and Pugh in [5].

DEFINITION. Let ε and T be any two continuous positive real valued functions on M, and let $p, q \in M$. An $(\varepsilon(x), T(x))$-*chain* for a flow ϕ on M from p to q is a finite sequence $\{(x_i, t_i)\}_{i=0}^n$ in $M \times \mathbb{R}$ such that

(1) $t_i \geq T(x_i),\ 0 \leq i \leq n$
(2) $d(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon(\phi_{t_i}(x_i)),\ 0 \leq i \leq n-1$
(3) $p = x_0,\ \ d(\phi_{t_n}(x_n), q) < \varepsilon(\phi_{t_n}(x_n))$

We say that a point $p \in M$ is *chain recurrent* for ϕ if for any continuous positive real valued functioins ε and T on M, there exists an $(\varepsilon(x), T(x))$- chain from p to itself. The set of all chain recurrent points of ϕ will be denoted by $CR(\phi)$. In [5], Peixoto and Pugh proved that a point $p \in M$ is chain recurrent if for any constant T and any continuous positive real valued function ε on M there exists an $(\varepsilon(x), T)$-chain from p to itself.

We say that a vector field X on M *generates* a flow ϕ on M if

$$\frac{\partial \phi(p,t)}{\partial t}|_{t=0} = X(p)$$

for all $p \in M$. The vector field $\dot{\phi}$ on M given by $\dot{\phi}(p) = \frac{\partial \phi(p,t)}{\partial t}|_{t=0}$ is said to be the *velocity vector field* of ϕ. We note that a vector field X on M fail to generate a flow on M, either because the differential equation $\frac{dx}{dt} = X$ may have not unique solutions, or because the solutions are not defined for all time.

Let $\mathcal{X}^0(M)$ be the set of all locally Lipschitz vector fields on M which generate flows on M with the C^0 Whitney topology: given a Riemannian metric $\|\|$ on M, a C^0 neighborhood of a vector field $X \in \mathcal{X}^0(M)$ will be specified by a positive continuous function $\varepsilon : M \to \mathbb{R}^+$, and will consist of all vector fields $Y \in \mathcal{X}^0(M)$ satisfying the pointwise estimate

$$\|X(p) - Y(p)\| < \varepsilon(p) \text{ for all } p \in M.$$

Let $\mathcal{F}(M)$ be the set of flows on M which can be generated by some vector field $X \in \mathcal{X}^0(M)$. The flows in $\mathcal{F}(M)$ will be topologized by means of their velocity vector fields. Thus a C^0 neighborhood of a flow $\phi \in \mathcal{F}(M)$ will consist of the flows ψ for which

$$\|\dot{\phi}(p) - \dot{\psi}(p)\| < \varepsilon(p) \text{ for all } p \in M$$

Although we are invoking a metric on M to define it, the C^0 Whitney topology on $\mathcal{F}(M)$ is independent of the metric we use.

We now state our main theorem which asserts that the Closing Lemma for flows $\phi \in \mathcal{F}(M)$ holds at every chain recurrent point of ϕ.

THEOREM. *(Closing Lemma for Chain Recurrence) Let p be a chain recurrent point of a flow $\phi \in \mathcal{F}(M)$, and let $\mathcal{U} \subset \mathcal{F}(M)$ be a neighborhood of ϕ in the C^0 Whitney topology. Then there exists $\psi \in \mathcal{U}$ such that ψ has a periodic orbit through p.*

Prior to the proof of the theorem, we give some remarks which make sense our results.

REMARKS. Until 1992, there was an open fundamental question in dynamics, reasonably called the C^1 Connecting Lemma: *for a flow ϕ on a manifold M and $p, q \in M$, we suppose*

$$\omega(p) \cap \alpha(q) \neq \emptyset$$

where $\alpha(q) = \{x \in M : \phi_{t_n}(q) \to x \text{ for some } t_n \to -\infty\}$. Does there exist a flow that C^1-approximates ϕ for which p, q lie on the same orbit?

Recently, Pugh gave an example to show that the C^1 Connecting Lemma is false. The example was constructed in the plane using the concept of the flow plug (for more details, see [7]). Also he constructed a C^1 flow ϕ on the punctured torus, $T^2 - \{0\}$, with prolongationally recurrent orbits which cannot be periodic using C^1 small perturbations (see figure 8 in [7]). Surprisingly, we can see that the flow ϕ on the punctured torus constructed by Pugh have the following properties:

(1) every fixed point of ϕ is hyperbolic,
(2) the point p is chain recurrent for ϕ, but
(3) the point p cannot be periodic under C^1 small perturbations of ϕ.

Consequently, we conclude that the C^1 Closing Lemma for chain recurrence in a noncompact surface does not hold even if every fixed point is hyperbolic. This contrasts with the case that the surface is $\mathbb{R}^2$ (or is planar). Then the C^r Closing Lemma for prolongational recurrence and even for chain recurrence is true (see [3],[5]).

In [2], we gave a characterization of the set of points which satisfy the Closing Lemma.

PROOF OF THE THEOREM. Let $\phi \in \mathcal{F}(M)$, $p \in CR(\phi)$ and ε a continuous positive real valued function on M. Suppose dim $M = n$, and let $Fix(\phi)$ be the set of all fixed points of ϕ. For each $x \in M - Fix(\phi)$, we choose a neighborhood U_x of x, $a_x > 0$ and a diffeomorphism $h_x : U_x \to \mathbb{R}^n$ such that

(1) $[-a_x, a_x]^n \subset h_x(U_x)$,
(2) $h_x(x) = 0$,
(3) $\dot{\phi} = Dh_x^{-1}(\frac{\partial}{\partial x^1})$ on U_x.

In this case the pair (U_x, h_x) is said to be a flow box for ϕ containing x. Let $I_x^n = [-a_x, a_x]^n$, $h_x(U_x) = B_x^n$ and

$$N_x = \sup_{p \in I_x^n} \sup\{\|Dh_x^{-1}(p)(v)\| : v \in T_p\mathbb{R}^n, \|v\| = 1\}$$

Then we have $N_x < \infty$. Let

$$k = \inf\{\varepsilon(p) : p \in h_x^{-1}(I_x^n)\}$$

Then we get $k > 0$. Choose $0 < b_x < a_x$ such that $\frac{nb_x}{a_x} < \frac{k}{N_x}$. Choose a continuous map $\delta : M \to \mathbb{R}^+$ such that

$$h_x(B_{\delta(x)}(x)) \subset (-a_x, a_x) \times (-b_x, b_x)^{n-1}$$

for each $x \in M - Fix(\phi)$. Since $p \in CR(\phi)$, we can select $(\delta(x), 1)$-chain $\{(x_i, t_i)\}_{i=0}^m$ for ϕ from p to p. Then we have

$$d(\phi_{t_i}(x_i), x_{i+1}) < \delta(\phi_{t_i}(x_i))$$

for $i = 0, 1, ..., m-1$. For each $0 \leq i \leq m$, we choose flow boxes (U_i, h_i) for ϕ at $\phi_{t_i}(x_i)$, where $U_i = B_{\delta(\phi_{t_i}(x_i))}(\phi_{t_i}(x_i))$; i.e. h_i maps U_i into $(-a_i, a_i) \times (-b_i, b_i)^{n-1}$, $\phi_{t_i}(x_i)$, $x_{i+1} \in U_i$ and $h_i(\phi_{t_i}(x_i)) = 0$. First we suppose that $U_i \cap U_j = \emptyset$ for $i \neq j$, and let

$$h_i(p_i) = h_i^{-1}(\{-a_i\} \times [-b_i, b_i]^{n-1}) \cap [x_i, \phi_{t_i}(x_i)]$$
$$h_i(q_i) = h_i^{-1}(\{-a_i\} \times [-b_i, b_i]^{n-1}) \cap [x_{i+1}, \phi_{t_{i+1}}(x_{i+1})],$$

where $[x, \phi_T(x)] = \{\phi_t(x) : 0 \leq t \leq T\}$ is an orbit segment of ϕ. Let $\beta_i : (-2, 2) \to B_i^n$ be a C^2 curve in B_i^n such that

$$\beta_i(-1) = p_i, \ \beta_i(0) = 0, \ \beta_i(1) = q_i \text{ and}$$
$$\|\dot{\beta}_i(t) - \frac{\partial}{\partial x_i}|_{\beta_i(t)}\| < \frac{nb_i}{a_i}$$

for $-1 < t < 1$,

$$\dot{\beta}_i(t) = \frac{\partial}{\partial x^1}|_{\beta_i(t)}$$

for $t \leq -1$ or $t \geq 1$. Let $\alpha_i = h_i^{-1} \circ \beta_i$. Then α_i is a C^2 curve in M such that

$$\begin{aligned}
\|\dot{\phi}(\alpha_i(t)) - \dot{\alpha}_i(t)\| &= \|Dh_i^{-1}(\beta_i(t))(\frac{\partial}{\partial x^1}|_{\beta_i(t)}) - Dh_i^{-1}(\beta_i(t))(\dot{\beta}_i(t))\| \\
&\leq \|Dh_i^{-1}(\beta_i(t))\| \|\frac{\partial}{\partial x^1}|_{\beta_i(t)} - \dot{\beta}_i(t)\| \\
&< N_i \frac{nb_i}{a_i} \leq N_i \frac{k}{N_i} \\
&\leq \varepsilon(\alpha_i(t)), \quad -1 < t < 1.
\end{aligned}$$

Even if the flow boxes are overlap, we can select a periodic curve α which we want. In this way, we can construct a periodic C^2 curve $\alpha : \mathbb{R} \to M$ such that

(1) $\alpha(0) = p = \alpha(T),\ \alpha(t+T) = \alpha(t)$ for some $T > 0$
(2) $\|\dot{\phi}(\alpha(t)) - \dot{\alpha}(t)\| < \varepsilon(\alpha(t)),\ \ t \in \mathbb{R}$.

Choose a tubular neighborhood U of the curve α in M. Extend $\dot{\alpha}$ to a C^1 vector field Y on U. Define a map $f : U \to \mathbb{R}$ by

$$f(x) = \|\dot{\phi}(x) - Y(x)\| - \varepsilon(x).$$

Then the map f is continuous. Since $\|\dot{\phi}(\alpha(t)) - Y(\alpha(t))\| < \varepsilon(\alpha(t))$ for $t \in \mathbb{R}$, we have

$$\alpha \subset \{x \in U : \|\dot{\phi}(x) - Y(x)\| < \varepsilon(x)\} \subset V,$$

where $\alpha = \{\alpha(t) : t \in \mathbb{R}\}$. Put $Y|_V = Y$. Then Y is a C^1 vector field on V such that $Y|_\alpha = \dot{\alpha}$ and $\|\dot{\phi}(x) - Y(x)\| < \varepsilon(x)$ for all $x \in V$. Choose a bump function $\lambda : M \to [0,1]$ such that

$$\lambda(x) = 0 \text{ if } \ x \notin V, \text{ and } \lambda(x) = 1 \text{ if } \ x \in \alpha.$$

Define a vector field $Z : M \to TM$ by

$$Z(x) = \lambda(x)Y(x) + (1 - \lambda(x))\dot{\phi}(x), \quad x \in M$$

Then Z is a locally Lipschitz vector field on M such that $Z = \dot{\phi}$ on $M - V$, and $Z = \dot{\alpha}$ on α. Since $Z = \dot{\phi}$ off a compact subset $\overline{V}$ of M and Z is locally Lipschitz, the vector field Z generates a flow on M, say ψ. Then α is a periodic orbit for ψ. Moreover we have $\psi \in \mathcal{U}_\varepsilon(\phi)$. In fact, for any $x \in V$, we get

$$\begin{aligned} \|\dot{\phi}(x) - \dot{\psi}(x)\| &= \|\dot{\phi}(x) - Z(x)\| \\ &\le |\lambda(x)| \|Y(x) - \dot{\phi}(x)\| \\ &\le \|Y(x) - \dot{\phi}(x)\| \\ &< \varepsilon(x). \end{aligned}$$

This completes the proof.

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