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ON A TOPOLOGICAL EXISTENCE THEOREM

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ABSTRACT. The purpose of this paper is to prove a new topological existence theorem by using the connectness property, and some applications are also given.

1. Introduction

In many problems of nonlinear functional analysis and applied mathematics, the maximal element technique is a very useful tool for proving the existence of solution. And we have already known many interesting maximal element existence theorems and their applications, e.g. see Browder [2], Fan [4], Mehta-Tarafdar [5], Tan-Yuan [7].

The classical maximal element existence theorem, due to Fan [4], is as follows: *Let X be a non-empty compact convex subset of a Hausdorff topological vector space E and let $T : X \to 2^X$ be a multimap such that for each $x \in X$, $T(x)$ is convex and $x \notin T(x)$, and for each $y \in X$, $T^{-1}(y)$ is open in X. Then there exists $\bar{x} \in X$ such that $T(\bar{x}) = \emptyset$.*

In the above, the convexity assumption is very essential and the main proving tool is the continuous selection technique. Still there have been many equivalent formulations of the above theorem, and

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also many generalizations and applications have been established, e.g. see [4,5,7,8].

In this paper, we shall give a new existence theorem by using the topological property of convex sets. We first prove a basic existence theorem by using the connectedness, and next prove a new fixed point theorem. Finally an equilibrium existence theorem and a variational inequality are established as applications.

2. Preliminaries

We first recall the following notations and definitions. Let A be a non-empty set. We shall denote by 2^A the family of all subsets of A. Let X, Y be non-empty topological spaces and $T : X \to 2^Y$ be a multimap. The multimap T is said to be *open* or have *open graph* if the graph of T ($\mathrm{Gr}\, T = \{(x, T(x)) \in X \times Y \,|\, x \in X\}$) is open in $X \times Y$. We may call $T(x)$ the *upper section* of T, and $T^{-1}(y)(= \{x \in X \,|\, y \in T(x)\})$ the *lower section* of T. It is easy to check that if T has open graph, then the upper and lower sections of T are open ; however the converse is not true in general. A multimap $T : X \to 2^Y$ is said to be *closed at* x if for each net $(x_\alpha) \to x$, $y_\alpha \in T(x_\alpha)$ and $(y_\alpha) \to y$, then $y \in T(x)$. A multimap T is *closed* if it is closed at every point of X. It is clear that if T is closed at x and $T(x_\alpha)$ is non-empty, then $T(x)$ is also non-empty. Note that if T is single-valued, then the closedness is equivalent to continuity as a function.

For a given multimap $T : X \to 2^Y$, $x \in X$ is called a *maximal element* for T if $T(x) = \emptyset$. Indeed, in real applications, the maximal element may be interpreted as the set of those objects in X that are the "best" or "largest" choices.

Finally we recall the following definitions of equilibrium theory in mathematical economics. Let I be a finite or an infinite set of agents. For each $i \in I$, let X_i be a non-empty set of actions. An *generalized* *game* $\Gamma = (X_i, A_i, P_i)_{i\in I}$ is defined as a family of ordered triples (X_i, A_i, P_i) where X_i is a non-empty topological space (a choice set), $A_i : \Pi_{j\in I} X_j \to 2^{X_i}$ is a constraint correspondence (multimap) and $P_i : \Pi_{j\in I} X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $\hat{x} \in X = \Pi_{i\in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. In particular, when $I = \{1, \cdots, n\}$, we may call Γ an *N-person game.*

3. Fixed Points and Some Applications

We begin with the following:

LEMMA. *Let X be a non-empty connected subset of a Hausdorff topological space E and $T : X \to 2^E$ be a closed multimap such that for each $y \in E$, $T^{-1}(y)$ is open (maybe empty) in X. If $T(x)$ is non-empty for some $x \in X$, then there exists non-empty subset $K \subset E$ such that $T(x) = K$ for each $x \in X$.*

PROOF. Suppose the assertion were false. Then we can find x_o, $x_1 \in X$ and $y_o \in E$ such that $y_o \in T(x_o) \setminus T(x_1)$. Since T is closed, the lower section $T^{-1}(y_o)$ is closed. In fact, for every net $(x_\alpha)_{\alpha\in\Gamma} \subset T^{-1}(y_o)$ with $(x_\alpha) \to x$, we have $y_o \in T(x_\alpha)$ for each $\alpha \in \Gamma$ and $(x_\alpha) \to x$, so by the closedness of T at x, $y_o \in T(x)$. Hence $x \in T^{-1}(y_o)$, so $T^{-1}(y_o)$ is closed. By the assumption, $T^{-1}(y_o)$ is also open. Since $x_o \in T^{-1}(y_o)$, by the connectedness of X, $T^{-1}(y_o) = X$. Therefore we have $y_o \in T(x)$ for each $x \in X$, which is a contradiction. Therefore T must be constant, i.e. there exists non-empty subset $K \subset E$ such that $T(x) = K$ for each $x \in X$.

Now we can prove the following fixed point theorem:

THEOREM 1. *Let X be a non-empty connected subset of a Hausdorff topological space E and $T : X \to 2^E$ be a closed multimap. If $T^{-1}(y_o)$ is non-empty open in X for some $y_o \in E$, then $y_o \in T(x)$*

for every $x \in X$. In particular, if $y_o \in X$, then y_o is a fixed point for T, i.e. $y_o \in T(y_o)$.

PROOF. Since T is closed, as in the proof of Lemma, the lower section $T^{-1}(y)$ is closed (maybe empty) for each $y \in E$. By the assumption, $T^{-1}(y_o)$ is both open and closed in X, and hence by the connectedness of X, $T^{-1}(y_o) = X$. Therefore $y_o \in T(x)$ for every $x \in X$. In particular, if $y_o \in X$, then we have $y_o \in T(y_o)$.

The above theorem can be useful in the following example:

EXAMPLE Let $X := \{(x,y) \in R^2 : y = \sin\frac{1}{x},\ x > 0\} \cup \{(0,0)\}$ be the topological sine curve in R^2 and a multimap $T : X \to 2^X$ be defined as follows:

$$T(x,y) = \begin{cases} X, & if\ \ (x,y) = (0,0), \\ \{(0,0)\}, & if\ \ (x,y) \neq (0,0). \end{cases}$$

Then T is clearly closed at every point in the connected set X and also $T^{-1}(0,0) = X$ is open. Therefore, by Theorem 1, we can find a fixed point for T. And it should be noted that the domain of T and each $T(x)$ are not convex, so that many known fixed point theorems (e.g. [2–5,8]) can not be suitable for T.

We can reformulate Theorem 1 to the following existence of maximal element:

COROLLARY. *Let X be a non-empty connected subset of a Hausdorff topological space E and $T : X \to 2^X$ be closed at every x, where $T(x) \neq \emptyset$, such that*

(1) *$T^{-1}(y)$ is open in X for each $y \in X$,*
(2) *$x \notin T(x)$ for each $x \in X$.*

Then T has a maximal element $\bar{x} \in X$, *i.e.* $T(\bar{x}) = \emptyset$.

PROOF. Suppose $T(x) \neq \emptyset$ for each $x \in X$. Then there exists $y \in T(z)$ for some $z \in X$, and so $z \in T^{-1}(y)$. By (1), $T^{-1}(y)$ is non-empty open in X. Hence, by Theorem 1, $y \in T(x)$ for each $x \in X$, and this contradicts the assumption (2). Therefore T has a maximal element.

As an application, we shall give a basic existence theorem of equilibrium for 1-person game:

THEOREM 2. *Let* $\Gamma = (X, A, P)$ *be an 1-person game such that*

(1) X *is a non-empty connected subset of a Hausdorff topological space,*
(2) *the correspondence* $A : X \to 2^X$ *is closed and* $A(x)$ *is non-empty connected for some* $x \in X$,
(3) *the correspondence* $P : X \to 2^X$ *is closed at every* x *where* $A(x) \cap P(x) \neq \emptyset$,
(4) *for each* $y \in X$, $A^{-1}(y)$ *is open in* X,
(5) *for each* $y \in X$, $P^{-1}(y)$ *is open in* X,
(6) *for each* $x \in \{x \in X : x \in A(x)\}$, $x \notin P(x)$.

Then Γ *has an equilibrium choice* $\hat{x} \in X$, *i.e.*

$$\hat{x} \in A(\hat{x}) \quad \textit{and} \quad A(\hat{x}) \cap P(\hat{x}) = \emptyset.$$

PROOF. By the assumptions (2) and (4), using Lemma, there exists a non-empty subset $K \subset X$ such that $A(x) = K$ for each $x \in X$. Then K is connected by the assumption (2) and the set $\{x \in X : x \in A(x)\}$ is exactly equal to K. We now consider a multimap $A \cap P : K \to 2^K$ defined by

$$(A \cap P)(x) := A(x) \cap P(x) \quad \text{for each } x \in K.$$

Then by the assumptions (2) - (6), the whole assumptions of the Corollary are satisfied, so that by applying Corollary to $A \cap P$, there exists some $\hat{x} \in K$ such that $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Therefore $\hat{x}$ is the desired equilibrium choice for Γ.

By modifying the method in [7], we can generalize Theorem 2 to N-person game or generalized games as in [7,8].

Finally, we can obtain the following basic variational inequality:

THEOREM 3. *Let X be a connected subset of a normed linear space E, E^* be its dual space and denote the dual pairing on $E^* \times E$ by $< w, x >$ for each $w \in E^*, x \in E$. Let $f : X \to E^*$ be a continuous map from the relative topology of X to the weak* topology of E^*. Suppose further that for every net $(x_\alpha) \subset X$, converging to $\bar{x}$, and $(y_\alpha) \subset X$, converging to $\bar{y}$ with $Re < f(x_\alpha), x_\alpha - y_\alpha > \; > \; 0$, we have $Re < f(\bar{x}), \bar{x} - \bar{y} > \; > \; 0$. Then there exists a point $\hat{x} \in X$ such that*

$$Re < f(\hat{x}), \hat{x} - x > \; \leq \; 0 \quad \textit{for all } x \in E.$$

PROOF. We first define a multimap $T : X \to 2^E$ by

$$T(x) := \{y \in E : Re < f(x), x - y > \; > \; 0\} \qquad \text{for each } x \in X.$$

Then $x \notin T(x)$ for each $x \in X$. We now claim that $T(\hat{x})$ is empty for some $\hat{x} \in X$. Suppose the contrary. Then, by Theorem 2.5.1 in [1] , the correspondence $\phi : x \to Re < f(x), x - y >$ is lower semicontinuous, so that each $T^{-1}(y) = \{x \in X : \phi(x) > 0\}$ is open, i.e. T has open lower sections. By the assumption, we can easily show that T is a closed multimap. Therefore, by Theorem 1, T has a fixed point, which is a contradiction. Hence we have that $T(\hat{x})$ is empty for some $\hat{x} \in X$. Therefore we have

$$Re < f(\hat{x}), \hat{x} - x > \; \leq \; 0 \quad \text{for all } x \in E.$$

It should be noted that Theorem 3 can be generalized to multimaps and non-compact sets.

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