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LIPSCHITZ STABILITY AND BOUNDEDNESS FOR F.D.E.

YOON HOE GU* AND HYUN SOOK RYU**

1. Basic Notions and Definitions

We consider a system of functional differential equations(F.D.E.)

$$x'(t) = f(t, x_t) \tag{N}$$

where $f : \mathbb{R}^+ \times \mathcal{C} \to \mathbb{R}^n$ is continuous, $f(t,0) = 0$, and $\mathcal{C} = C([-r,0], \mathbb{R}^n), r > 0$.

For any $x_t \in \mathcal{C}, x_t(\theta) = x(t+\theta)$ for all $\theta \in [-r,0]$ and $||x_t|| = \sup_{\theta\in[-r,0]} |x_t(\theta)|$, where $|\cdot|$ denotes an arbitrary vector norm in $\mathbb{R}^n$. Let $x = x(t_0, \phi, f)$ be the unique solution of (1) with initial function ϕ such that $x_{t_0} = \phi$.

The value of $x(t_0, \phi, f)$ at t will be $x(t) = x(t_0, \phi, f)(t)$.

DEFINITION 1. The trivial solution $x = 0$ of (N) is said to be *uniformly Lipschitz stable* (ULS) if for any $t_0 \geq 0$, there exists a $\delta > 0$ such that if $||\phi|| < \delta$, then $|x_t(t_0, \phi, f)| < M||\phi||$ for $t \geq t_0$.

DEFINITION 2. The solutions are *uniformly bounded*(UB) if , for any $\alpha > 0$, there is a $\beta = \beta(\alpha) > 0$ such that for all $t_0 \geq 0, \phi \in \mathcal{C}$ with $||\phi|| \leq \alpha$, we have $|x(t_0, \phi, f)(t)| \leq \beta$ for all $t \geq t_0$.

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2. Lipschitz Stability and Boundedness

LEMMA 2.1 [3, page 46]. *Assume that $f \in C[\mathbb{R} \times \mathcal{C}, \mathbb{R}^n]$ possesses continuous partial derivatives $\frac{\partial f}{\partial \phi}$ on $\mathbb{R} \times \mathcal{C}$. Let the solution $x(t_0, \phi, f)(t)$ of (N) exist for $t \geq t_0$. Then $D_\phi x(t_0, \phi, f)(t)$ is a linear operator from $\mathcal{C}$ to $\mathbb{R}^n$, $D_\phi x(t_0, \phi, f)(t_0) = I$, the identity, and $D_\phi x(t_0, \phi, f)\psi(t)$, for each ψ in $\mathcal{C}$, is the solution of the variational equation*

$$\text{(Z)} \qquad z'(t) = D_\phi f(t, x_t(t_0, \phi, f))z_t.$$

In particular $D_\phi x(t_0, 0, f)\psi(t)$ satisfy the equation

$$\text{(V)} \qquad v'(t) = D_\phi f(t, x_t(t_0, 0, f))v_t,$$

where $D_\phi = \frac{\partial}{\partial \phi}$ is the derivative operator.

LEMMA 2.2 [2]. *Assume that $x(t_0, \psi_1, f)(t)$ and $x(t_0, \psi_2, f)(t)$ are the solutions of (N) through (t_0, ψ_1) and (t_0, ψ_2), respectively, which exist for $t \geq t_0$ and such that ψ_1 and ψ_2 belong to a convex subset D of $\mathcal{C}$. Then for $t \geq t_0$,*

$$\begin{aligned} &x(t_0, \psi_1, f)(t) - x(t_0, \psi_2, f)(t) \\ &= \int_0^1 \left[D_\phi x(t_0, s\psi_1 + (1-s)\psi_2)(t)ds\right] \cdot (\psi_1 - \psi_2). \end{aligned}$$

PROOF. Define $X(s) = x(t_0, s\psi_1 + (1-s)\psi_2)(t), 0 \leq s \leq 1$. Then

$$\frac{d}{ds}X(s) = D_\phi x(t_0, s\psi_1 + (1-s)\psi_2)(t) \cdot (\psi_1 - \psi_2)$$

and $X(1) = x(t_0, \psi_1, f)(t)$, $X(2) = x(t_0, \psi_2, f)(t)$. By integrating the above equation from 0 to 1, we have the conclusion.

THEOREM 2.3. *If $|D_\phi x(t_0,\phi,f)(t)| \le M$ whenever $\|\phi\| < \delta$ for some $\delta > 0$, and for all $t \ge t_0$, then the zero solution of (N) is ULS.*

PROOF. Since $x(t_0,0,f) \equiv 0$, we have

$$x(t_0,\phi,f)(t) = \int_0^1 D_\phi x(t_0,s\phi,f)ds \cdot \phi,$$

by Lemma 2.2. Hence

$$\begin{aligned}|x_t(t_0,\phi,f)| &\le \int_0^1 |D_\phi x(t_0,s\phi,f)|ds \cdot \|\phi\| \\ &\le M\|\phi\|,\end{aligned}$$

for all $t \ge t_0$ and $\|\phi\| \le \delta$.

THEOREM 2.4. *The zero solution of (V) is ULS if and only if $D_\phi x(t_0,0,f)$ is UB.*

PROOF. Note that the solution of (V) through (t_0,ψ) has the form

$$v = D_\phi x(t_0,0,f)\psi = x(t_0,\psi,f) - x(t_0,0,f) - R(0,\psi),$$

where $R(0,\psi) \equiv 0(\|\psi\|), x(t_0,0,f) = 0$. Hence there exists a constant $\alpha > 0$ such that

$$|D_\phi x(t_0,0,f)(t)| \le \beta(\alpha) \qquad \text{for } \|\psi\| < \alpha \text{ and } t \ge t_0.$$

This implies that $D_\phi x(t_0,0,f)$ is UB.

The converse is trivial from the definition.

COROLLARY 2.5. *If the system (N) is ULS, then there exists a constant $\delta > 0$ such that*

$$|D_\phi x(t_0, 0, f)(t)| \leq M(\alpha) \qquad \text{for } ||\psi|| < \alpha \text{ and } t \geq t_0.$$

PROOF. If the system (N) is ULS, then the zero solution of (V) is ULS[2]. This Corollary follows from Theorem 2.4.

REMARK 1. If the zero solution of (N) is uniformly Lipschitz stable, then the zero solution of (N) is uniformly stable. The converse is not true in general[1]. But, for the linear F.D.E., they are equivalent[2].

Now, we consider the homogeneous linear functional differential equation

$$\text{(LH)} \qquad x'(t) = L(t, x_t).$$

Assume that there is an $n \times n$ matrix function $\eta(t, \theta)$ such that

$$\eta(t,\theta) = 0, \quad \text{for} \quad \theta \geq 0,$$
$$\eta(t,\theta) = \eta(t, -r), \quad \text{for} \quad \theta \leq -r,$$

where $\eta(t,\theta)$ is measurable in $(t,\theta) \in \mathbb{R} \times \mathbb{R}$, continuous from the left in θ on $(-r, 0)$ has bounded variation in θ on $[-r, 0]$ for each t. Moreover, suppose that there is an $m \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$ such that

$$\text{Var}_{[-r,0]}\eta(t, \cdot\,) \leq m(t)$$

and

$$L(t,\phi) = \int_{-r}^{0} d[\eta(t,\theta)]\phi(\theta),$$

for all $t \in \mathbb{R}$ and $\phi \in \mathcal{C}$.

Obviously, $|L(t,\phi)| \leq m(t)||\phi||$ [3].

DEFINITION 2.6. The *fundamental matrix solution* $U(t,s)$ of (LH) is the solution of the equations,

$$\frac{\partial U(t,s)}{\partial t} = L(t, U_t(\cdot\,, t_0)), \qquad t \geq s \geq t_0 \text{ a.e. in } s \text{ and } t$$

where

$$U(t,s) = \begin{cases} 0, & \text{for} \quad s-r \leq t < s \\ I, & \text{for} \quad t = s, \end{cases}$$

and $U_t(\cdot\,, s)(\theta) = U(t+\theta, s), -r \leq \theta \leq 0$.

LEMMA 2.7 (Variation of constants formula)[3].

If L satisfies the above hypotheses and $x(t_0,\phi,h)$ is the solution of the nonhomogeneous system

$$\text{(NH)} \qquad x'(t) = L(t, x_t) + h(t), \qquad t \geq t_0$$
$$x_{t_0} = \phi,$$

then

$$(*) \qquad x(t_0,\phi,h)(t) = x(t_0,\phi,0)(t) + \int_{t_0}^{t} U(t,s)h(s)ds, \quad t \geq t_0$$
$$x_{t_0} = \phi,$$

where $x(t_0,\phi,0)$ is the solution of (LH).

If $x_t(t_0,\phi,0) \equiv T(t,t_0)\phi$, then $T(t,t_0) : \mathcal{C} \to \mathcal{C}$ is a continuous linear operator. So the relation $(*)$ has the equivalent form:

$$(**) \qquad x_t(t_0,\phi,h) = T(t,t_0)\phi + \int_{t_0}^{t} T(t,s)X_0 h(s)ds, \qquad t \geq t_0,$$

where

$$X_0(\theta) = \begin{cases} 0 & -r \leq \theta < 0, \\ I & \theta = 0. \end{cases}$$

THEOREM 2.8. *The zero solution of (LH) is ULS if and only if there exists $K > 0$ such that*

$$\text{(T)} \qquad |T(t, t_0)| \le K \qquad \text{for } t \ge t_0.$$

PROOF. If the zero solution of (LH) is ULS, then there exists $\delta > 0, M > 0$ such that

$$|x_t(t_0, \phi, 0)| = |T(t, t_0)\phi| \le M||\phi||$$

for $t \ge t_0$ and $||\phi|| < \delta$. Thus $|T(t, t_0)| \le M$ for $t \ge t_0$.

Conversely, the zero solution of (LH) is uniformly stable [3, page 163] and uniformly Lipschitz stable [2, Theorem 1].

COROLLARY 2.9. *If there is a constant m_1 such that*

$$\text{(H)} \qquad \int_t^{t+r} m(u)du \le m_1 \qquad \text{for } t \in \mathbb{R},$$

then the zero solution of (LH) is ULS if and only if there is a constant K such that

$$\text{(U)} \qquad |U(t, s)| \le K \quad \text{for } t \ge s \text{ and for all } s \in \mathbb{R}.$$

PROOF. With hypotheses (H), (T) and (U) are equivalent[3]. By Theorem 2.8, The proof is completed.

COROLLARY 2.10. *Suppsoe that the Hypotheses (T) and*

$$\int_{t_0}^{\infty} h(s)ds < \infty$$

are satisfied. Then the zero solution of (NH) is UB.

PROOF. By the variation of constants formulas(**),

$$\begin{aligned}|x_t| &\leq K\big(||\phi|| + \int_{t_0}^{t} h(s)ds\big)\\ &\leq \beta(\alpha)\end{aligned}$$

for $||\phi|| \leq \alpha$ and for all $t \geq t_0$.

Let $f(t,x_t) = D_\phi f(t,0)x_t + h(t,x_t)$ and $D_\phi f(t,0)x_t \equiv L(t,x_t)$.

THEOREM 2.11. *Suppose that the hypothesis (T) and*

$$|h(t,x_t)| \leq \varepsilon|x_t|, \qquad \varepsilon \leq \frac{1}{K}.$$

Then the zero olution of (N) is ULS.

PROOF. By variation of constants formula(**),

$$x_t(t_0,\phi,f) = T(t,t_0)\phi + \int_{t_0}^{t} T(t,s)X_0 h(s,x_s)ds.$$

Therefore

$$\begin{aligned}|x_t| &\leq K||\phi|| + K\varepsilon \int_{t_0}^{t} |x_s|ds\\ &\leq K||\phi|| + K\varepsilon \sup_{t_0 \leq s \leq t} |x_s| \qquad \text{for } t \geq t_0 \geq 0.\end{aligned}$$

Thus $\sup_{t_0 \leq s \leq t} |x_s| \leq \frac{K}{1-\varepsilon K}||\phi|| \leq M||\phi||$. This completes the proof of the theorem.

REMARK 2. With (U) ,(H) and $\varepsilon \leq \frac{1}{K}$, the zero solution of (N) is ULS.

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DEPARTMENT OF MATHEMATICS
HANSEO UNIVERSITY
SEOSAN 352-820, KOREA
**
DEPARTMENT OF MATHEMATICS
CHUNGNAM NATIONAL UNIVERSITY
TAEJON 305-764, KOREA