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FUNDAMENTAL MATRICES OF THE VARIATIONAL SYSTEMS FOR THE NONLINEAR SYSTEMS WITH A SMALL PARAMETER

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ABSTRACT. We show that $\frac{\partial x}{\partial \gamma}(t,\tau,\gamma,\lambda,\varepsilon)$ is a fundamental matrix of the variational system $\dot{y} = f_x(t, x(t,\tau,\gamma,\lambda,\varepsilon),\lambda,\varepsilon)y$ corresponding to the solution $x(t,\tau,\gamma,\lambda,\varepsilon)$ of $\dot{x} = f(t,x,\lambda,\varepsilon)$.

1. Introduction

Basti [1] investigated the asymptotic equivalence (for the definition, see [1]) between two nonlinear parametric systems with a small parameter

$$\dot{x} = f(t,x,\lambda,\varepsilon), \qquad \cdot = \frac{d}{dt} \tag{1}$$

and

$$\dot{y} = f(t,y,\psi(t,\varepsilon),\varepsilon) + g(t,y,\varepsilon). \tag{2}$$

Here f and the Jacobian matrices $f_x = \frac{\partial f}{\partial x}, f_\lambda = \frac{\partial f}{\partial \lambda}$ are continuous for $(t,x,\lambda,\varepsilon)$ in $\mathbb{R}^+ \times \mathbb{R}^n \times S_c \times (0,\varepsilon_1]$ into $\mathbb{R}^n$, S_c is the closed ball of radius c in $\mathbb{R}^m$, m and n are positive integers, $\mathbb{R}^+ = [0,\infty)$, $\psi(t,\varepsilon)$ is a continuously differentiable function with respect to t for $(t,\varepsilon) \in \mathbb{R}^+ \times (0,\varepsilon_1]$ into S_c and g is a continuous $\mathbb{R}^n$-valued function defined on

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$\mathbb{R}^+ \times \mathbb{R}^n \times (0, \varepsilon_1]$. Furthermore, Voskresenskii [3] studied the asymtotic equivalence between two systems

$$\dot{x} = f(t, x, \lambda^*(\varepsilon), \varepsilon) \tag{3}$$

and

$$\dot{y} = f(t, y, \psi(t, \varepsilon), \varepsilon) + g(t, y, \psi(t, \varepsilon), \varepsilon), \tag{4}$$

where

$$\lambda^*(\varepsilon) = \lim_{k \to \infty} \psi(s_k, \varepsilon)$$

for some sequence (s_k) with $s_k \to \infty$ as $k \to \infty$.

To obtain their results they needed the fundamental matrices

$$\frac{\partial x}{\partial \gamma}(t, \tau, \gamma, \lambda, \varepsilon) \qquad \text{and} \qquad \frac{\partial x}{\partial \lambda}(t, \tau, \gamma, \lambda, \varepsilon) \tag{5}$$

which satisfy the variational systems

$$\dot{y} = f_x(t, x(t, \tau, \gamma, \lambda, \varepsilon), \lambda, \varepsilon) y \tag{6}$$

and

$$\dot{y} = f_x(t, x(t, \tau, \gamma, \lambda, \varepsilon), \lambda, \varepsilon) y + f_\lambda(t, x(t, \tau, \gamma, \lambda, \varepsilon), \lambda, \varepsilon) \tag{7}$$

respectively, where $x(t, \tau, \gamma, \lambda, \varepsilon)$ is the solution of (1) satisfying $x(\tau) = \gamma$ for every $\tau \in \mathbb{R}^+, \gamma \in \mathbb{R}^n, \lambda \in S_c$ and $\varepsilon \in (0, \varepsilon_1]$.

The fundamental matrices (5) have the property that

$$\frac{\partial x}{\partial \gamma}(\tau, \tau, \lambda, \varepsilon) = I \text{ (identity matrix)}$$

and

$$\frac{\partial x}{\partial \lambda}(\tau, \tau, \gamma, \lambda, \varepsilon) = 0.$$

Moreover

$$\frac{\partial x}{\partial \tau}(t, \tau, \gamma, \lambda, \varepsilon) = -\frac{\partial x}{\partial \gamma}(t, \tau, \gamma, \lambda, \varepsilon) f(\tau, \gamma, \lambda, \varepsilon) \tag{8}$$

for every $(\tau, \gamma, \lambda, \varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^n \times S_c \times (0, \varepsilon_1]$.

In this note we show that $\frac{\partial x}{\partial \gamma}(t, \tau, \gamma, \lambda, \varepsilon)$ is a fundamental matrix of the variational system (6) corresponding to the solution $x(t, \tau, \gamma, \lambda, \varepsilon)$ and derive the formula (8). Thus this note is a supplement of [1].

The symbol $|.|$ will be need to denote any convenient vector norm in $\mathbb{R}^n$.

2. Main Results

The following is the generalization of the well-known Gronwall's inequality.

LEMMA 1. *Let m, n, u be real continuous functions defined on $[a, b]$ with $u(t) > 0$. If*

$$m(t) \le n(t) + \int_a^t u(s) m(s) ds$$

on $[a, b]$, then

$$m(t) \le n(t) + \int_a^t u(s) n(s) \exp[\int_s^t u(\tau) d\tau] ds$$

on $[a, b]$. Moreover, if n is constant, then

$$m(t) \le n \exp[\int_a^t u(\tau) d\tau]$$

whenever

$$m(t) \le n + \int_a^t u(s) m(s) ds.$$

LEMMA 2. *Suppose that $f(t,x,\lambda,\epsilon)$ is a Lipschitz function with respect to x with a Lipschitz constant $k(t)$. Then we have*

$$|x(t,\tau,\mu,\lambda,\varepsilon)-x(t,\tau,\gamma,\lambda,\varepsilon)| \le |\mu-\gamma|\exp[\int_\tau^t k(s)ds]. \tag{9}$$

PROOF.

$$\begin{aligned}
&|x(t,\tau,\mu,\lambda,\varepsilon)-x(t,\tau,\gamma,\lambda,\varepsilon)|\\
&\le |\mu-\gamma|+\int_\tau^t |f(s,x(s,\tau,\mu,\lambda,\varepsilon),\lambda,\varepsilon)-f(s,x(s,\tau,\gamma,\lambda,\varepsilon),\lambda,\varepsilon)|ds\\
&\le |\mu-\gamma|+\int_\tau^t k(s)|x(s,\tau,\mu,\lambda,\varepsilon)-x(s,\tau,\gamma,\lambda,\varepsilon)|ds.
\end{aligned}$$

Thus the inequality follows from Lemma 1.

THEOREM 3. *If $f(t,x,\lambda,\varepsilon)$ is differentiable in a domain $D\subset\mathbb{R}^+\times\mathbb{R}^n\times S_c\times(0,\varepsilon_1]$ for $t\in\mathbb{R}^+$, then $x(t,\tau,\gamma,\lambda,\varepsilon)$ is differentiable with respect to γ and $\frac{\partial x}{\partial\gamma}(t,\tau,\gamma,\lambda,\varepsilon)$ is a fundamental matrix of (6).*

PROOF. Let μ be such that $x(t,\tau,\mu,\lambda,\varepsilon)$ is in the domain of definition and $\Phi(t,\tau,\lambda,\varepsilon)$ be a fundamental matrix of (6) with $\Phi(\tau,\tau,\lambda,\varepsilon)=I$. Then $\Phi(t,\tau,\lambda,\varepsilon)(\mu-\gamma)$ becomes a solution of (6) which for $t=\tau$ coincides with $\mu-\gamma$. We show that

$$|x(t,\tau,\mu,\lambda,\varepsilon)-x(t,\tau,\gamma,\lambda,\varepsilon)-\Phi(t,\tau,\lambda,\varepsilon)(\mu-\gamma)| = o(|\mu-\gamma|). \tag{10}$$

$$\begin{aligned}
&x(t,\tau,\mu,\lambda,\varepsilon)-x(t,\tau,\gamma,\lambda,\varepsilon)-\Phi(t,\tau,\lambda,\varepsilon)(\mu-\gamma)\\
&=\mu+\int_\tau^t f(s,x(s,\tau,\mu,\lambda,\varepsilon),\lambda,\varepsilon)ds-\gamma-\int_\tau^t f(s,x(s,\tau,\gamma,\lambda,\varepsilon),\lambda,\varepsilon)ds\\
&\quad-(\mu-\gamma)-\int_\tau^t \frac{\partial f}{\partial x}(s,x(s,\tau,\gamma,\lambda,\varepsilon),\lambda,\varepsilon)\Phi(s,\tau,\lambda,\varepsilon)(\mu-\gamma)ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_\tau^t \Big\{ [f(s, x(s,\tau,\mu,\lambda,\varepsilon),\lambda,\varepsilon) - f(s, x(s,\tau,\gamma,\lambda,\varepsilon),\lambda,\varepsilon)] \\
&\quad - \frac{\partial f}{\partial x}(s, x(s,\tau,\gamma,\lambda,\varepsilon),\lambda,\varepsilon)\Phi(s,\tau,\lambda,\varepsilon)(\mu-\gamma)\Big\} ds \\
&= \int_\tau^t f_x(s, x(s,\tau,\gamma,\lambda,\varepsilon),\lambda,\varepsilon)[x(s,\tau,\mu,\lambda,\varepsilon) - x(s,\tau,\gamma,\lambda,\varepsilon) \\
&\quad - \Phi(s,\tau,\lambda,\varepsilon)(\mu-\gamma)]ds + \int_\tau^t o(|x(s,\tau,\mu,\lambda,\varepsilon) - x(s,\tau,\gamma,\lambda,\varepsilon)|)ds
\end{aligned}
$$

by the differentiability of $f(t,x,\lambda,\varepsilon)$ in a domain D. By Lemma 2, we have

$$
|x(s,\tau,\mu,\lambda,\varepsilon) - x(s,\tau,\gamma,\lambda,\varepsilon)| = o(|\mu-\gamma|) \quad \text{for fixed } t.
$$

Therefore

$$
\begin{aligned}
&|x(t,\tau,\mu,\lambda,\varepsilon) - x(t,\tau,\gamma,\lambda,\varepsilon) - \Phi(t,\tau,\lambda,\varepsilon)|(\mu-\gamma)| \\
\le &\int_\tau^t k(s)|x(s,\tau,\mu,\lambda,\varepsilon) - x(s,\tau,\gamma,\lambda,\varepsilon) - \Phi(s,\tau,\lambda,\varepsilon)(\mu-\gamma)|ds \\
&+ o(|\mu-\gamma|).
\end{aligned}
$$

Now, in view of Lemma 1, we have

$$
\begin{aligned}
&|x(t,\tau,\mu,\lambda,\varepsilon) - x(t,\tau,\gamma,\lambda,\varepsilon) - \Phi(t,\tau,\lambda,\varepsilon)(\mu-\gamma)| \\
&\le o(|\mu-\gamma|) + \int_\tau^t k(s)o(|\mu-\gamma|)\exp[\int_s^t k(u)du]ds.
\end{aligned}
$$

Hence (10) holds. This completes the proof.

THEOREM 4. *The formula (8) holds, i.e.,*

$$
\frac{\partial x}{\partial \tau}(t,\tau,\gamma,\lambda,\varepsilon) = -\frac{\partial x}{\partial \gamma}(t,\tau,\gamma,\lambda,\varepsilon)f(\tau,\gamma,\lambda,\varepsilon).
$$

PROOF. We define

$$\hat{x}_h(t) = \frac{1}{h}[x(t,\tau+h,\gamma,\lambda,\varepsilon) - x(t,\tau,\gamma,\lambda,\varepsilon)].$$

Note that

$$x(t,\tau+h,\gamma,\lambda,\varepsilon) = x(t,\tau,x(\tau,\tau+h,\gamma,\lambda,\varepsilon),\lambda,\varepsilon)$$

and

$$x(\tau,\tau+h,\gamma,\lambda,\varepsilon) \to x(\tau,\tau,\gamma,\lambda,\varepsilon) = \gamma \qquad \text{as } h \to 0.$$

Then

$$h\hat{x}_h(t) = [\frac{\partial x}{\partial \gamma}(t,\tau,\gamma,\lambda,\varepsilon) + o(1)][x(\tau,\tau+h,\gamma,\lambda,\varepsilon) - \gamma] \quad \text{as } h \to 0.$$

By the mean value theorem, there exist $\theta = \theta_k, k = 1,\ldots,n$, with $0 < \theta < 1$ such that

$$\begin{aligned} &x_k(\tau,\tau+h,\gamma,\lambda,\varepsilon) - x_k(\tau,\tau,\gamma,\lambda,\varepsilon) \\ = &- hf_k(\tau+\theta h, x(\tau+\theta h,\tau+h,\gamma,\lambda,\varepsilon),\lambda,\varepsilon). \end{aligned}$$

For each k, we have

$$f_k(\tau+\theta h, x(\tau+\theta h,\tau+h,\gamma,\lambda,\varepsilon),\lambda,\varepsilon) = f_k(\tau,\gamma,\lambda,\varepsilon)+o(1) \qquad \text{as } h \to 0.$$

Hence

$$\hat{x}_h(t) = [-\frac{\partial x}{\partial \gamma}(t,\tau,\gamma,\lambda,\varepsilon) + o(1)]f(\tau,\gamma,\lambda,\varepsilon) + o(1).$$

It follows that

$$\frac{\partial x}{\partial \tau}(t,\tau,\gamma,\lambda,\varepsilon) = \lim_{h\to 0} \hat{x}_h(t) = -\frac{\partial x}{\partial \gamma}(t,\tau,\gamma,\lambda,\varepsilon)f(\tau,\gamma,\lambda,\varepsilon).$$

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