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STABILITIES FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Using the comparison principle and inequalities we obtain some results on boundedness and stabilities of solutions of the nonlinear functional differential equation $y' = f(t,y) + g(t,y,Ty)$.

1. Introduction

We consider the nonlinear functional differential equation

$$y' = f(t,y) + g(t,y,Ty), \tag{1}$$

where $t \in \mathbb{R}^+ = [0,\infty)$, $x \in \mathbb{R}^n$, $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) \equiv 0$, the derivative $f_x \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t,0,0) \equiv 0$ and T is a continuous operator mapping from $C(\mathbb{R}^+, \mathbb{R}^n)$ into $C(\mathbb{R}^+, \mathbb{R}^n)$. Equation (1) can be considered as the perturbed equation of

$$x' = f(t,x) \tag{2}$$

and may represent several interesting cases, namely, integrodifferential equations and retarded functional differential equations etc.[3].

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Let $x(t) = x(t, t_0, x_0)$ be the unique solution of (2) passing through $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ and $\Phi = \Phi(t, t_0, x_0)$ the fundamental matrix of the associated variational equation

$$(3) \qquad z'(t) = f_x(t, x(t, t_0, x_0))z(t)$$

such that $\Phi(t_0, t_0, x_0)$ is the identity matrix (see [5]).

Pachpatte[6] studied the asymptotic behavior of solutions of Equation (1) under suitable conditions on the perturbing term g and the operator T. Also, Pinto[3] proved theorems which relate the asymptotic behavior and boundedness of the solutions of Equations (1) and (2).

In this paper we obtain some results on boundedness and basic stability properties of solutions of Equation (1) under suitable conditions on g and Φ. To do this we need some differential and integral inequalities.

We assume that for any two continuous functions $u, v \in C(\mathbb{R}^+, \mathbb{R}^+)$ the continuous operator T satisfies the following property :

$$u(t) \leq v(t), \quad 0 \leq t \leq t_1, t_1 \in \mathbb{R}^+$$

implies

$$Tu(t) \leq Tv(t), \quad 0 \leq t \leq t_1$$

and

$$|Tu| \leq T|u|.$$

2. Boundedness and Stabilities

The following lemma 2.1 is an adaptation of Theorem 5.1.1[5].

LEMMA 2.1. *Suppose that* $W(t,u,v) \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ *is monotone nondecreasing in* u *and* v *for each fixed* $t \in \mathbb{R}^+$ *with the property that*

$$m(t) - \int_{t_0}^{t} W(s, m(s), Tm(s))ds < u(t) - \int_{t_0}^{t} W(s, u(s), Tu(s))ds,$$

for $t \geq t_0 \geq 0$ *and* $m, u \in C(\mathbb{R}^+)$. *If* $m(t_0) < u(t_0)$, *then* $m(t) < u(t)$ *for all* $t \geq t_0 \geq 0$.

PROOF. Assume that there exists a $t_1 > t_0$ such that

$$m(t_1) = u(t_1) \quad \text{and} \quad m(t) < u(t), \qquad t_0 \leq t < t_1.$$

Then we have

$$Tm(t) \leq Tu(t), \qquad t_0 \leq t \leq t_1$$

and thus

$$\begin{aligned} m(t_1) &< u(t_1) - \int_{t_0}^{t_1} W(s, u(s), Tu(s))ds + \int_{t_0}^{t_1} W(s, m(s), Tm(s))ds \\ &\leq u(t_1). \end{aligned}$$

This contradicts the fact that, at $t = t_1, m(t_1) = u(t_1)$ and hence the proof is complete.

A function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is said to belong to the *class* H if

(i) $w(u)$ is nondecreasing and continuous for $u \geq 0$ and positive for $u > 0$,

(ii) there exists a function ϕ which is continuous on $\mathbb{R}^+$ with $w(\alpha u) \leq \phi(\alpha) w(u)$ for $\alpha > 0, u \geq 0$.

LEMMA 2.2 [2]. *Let $u, a, b, c, \in C(\mathbb{R}^+), w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that for some $d > 0$,*

(4)
$$u(t) \le d + \int_{t_0}^{t} a(s)w(u(s))ds + \int_{t_0}^{t} b(s)\int_{t_0}^{s} c(\tau)w(u(\tau))d\tau ds, \quad t \ge t_0 \ge 0.$$

Then

$$(5)\quad u(t) \le W^{-1}[W(d) + \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau)ds],\ t_0 \le t < b_1,$$

where $W(u) = \int_{u_0}^{u}\frac{dz}{w(z)}, u_0 > 0$ and $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\{t \ge t_0 : W(d) + \int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau)ds \in \mathrm{Dom}W^{-1}\}.$$

THEOREM 2.3. *Suppose that*

$$|\Phi(t,s,y)g(s,y,Ty)| \le a(s)w(|y|) + b(s)|Ty|, \quad t,s \in \mathbb{R}^+,$$

where $a, b \in C(\mathbb{R}^+), a, b \in L_1(I)$ and $I = [t_0,\infty)$. Further, suppose that the operator T satisfies the inequality

$$|Ty(t)| \le \int_{t_0}^{t} c(s)w(|y(s)|)ds,$$

where $c \in C(R^+)$ and $c \in L_1(I)$,

$$M(t_0) = W^{-1}[W(d) + \int_{t_0}^{\infty}(a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau)ds] < \infty,$$
$$W(\infty) = \infty.$$

Then for every bounded solution $x(t) = x(t, t_0, x_0)$ of (2) on $I = [t_0, \infty)$, the corresponding solution $y(t) = y(t, t_0, x_0)$ of (1) is also bounded.

PROOF. We obtain

$$|y(t)| \leq W^{-1}[W(d) + \int_{t_0}^{t} (a(s) + b(s) \int_{t_0}^{s} c(\tau)d\tau)ds] \leq M(t_0),\ t \geq t_0,$$

where d is the bound for $|x(t)|$, by an application of Lemma 2.2. The theorem is proved.

We now give stability definitions of Equation (1).

DEFINITION. Equation (1)(or the trivial solution $y = 0$ of (1)) will be called

(S) *stable* if for any two solutions of $y(t, t_0, y_0)$ and $z(t, t_0, z_0)$ of (1) with the initial conditions $y(t_0) = y_0$ and $z(t_0) = z_0$, respectively,

$$|y_0 - z_0| < \delta$$

implies

$$|y(t, t_0, y_0) - z(t, t_0, z_0)| < c\delta, (c = \text{const}), \text{for } \delta > 0 \text{ and } t \geq t_0,$$

(US) *uniformly stable* if the δ in (S) is independent of the time t_0,

(AS) *asymptotically stable* if it is stable and

$$\lim_{t \to \infty} |y(t, t_0, y_0) - z(t, t_0, z_0)| = 0,$$

(UAS) *uniformly asymptotically stable* if it is uniformly stable and for each $\epsilon > 0$, there are $\delta > 0$ and $T(\epsilon)$ such that $|y_0 - z_0| < 0, t_0 \geq 0$ implies

$$|y(t) - z(t)| < \epsilon, \quad t \geq t_0 + T(\epsilon).$$

THEOREM 2.4. *Suppose that*

$$|\Phi(t,t_0,x_0)| \le a(|x_0|), \quad t \ge t_0,$$

where $a \in C(\mathbb{R}^+)$ *and* $a(u)$ *is nondecreasing in* u*;*

$$\begin{aligned}&|\Phi(t,s,y(s))g(s,y(s),Ty(s)) - \Phi(t,s,z(s))g(s,z(s),Tz(s))|\\ &\le a(|y(s)-z(s)|)W(s,|y(s)-z(s)|,T|y(s)-z(s)|), \quad t,s \in \mathbb{R}^+\end{aligned}$$

where $W(t,u,v)$ *is the same function as defined in Lemma 2.1. Then the stability (asymptotic stability) of Equation (1) follows from the stability (asymptotic stability) of the scalar differential equation*

$$\text{(S-1)} \qquad u' = a(u)W(t,u,Tu), \quad u(t_0) = u_0, \quad t \ge t_0,$$

where $a(|y_0| + |z_0|) \ge 1$ *and* $W(t,0,0) = 0$.

PROOF. Using the nonlinear variation of parameters formula[4, Theorem 2.1.3-5], any two solutions of (1) passing through (t_0, x_0) and (t_0, z_0) are represented by

$$y(t) = y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,s,y(s))g(s,y(s),Ty(s))ds,$$

$$z(t) = z(t,t_0,z_0) = \bar{x}(t,t_0,z_0) + \int_{t_0}^{t} \Phi(t,s,z(s))g(s,z(s),Tz(s))ds.$$

Letting $m(t) = |y(t) - z(t)|$, we obtain

$$\begin{aligned}|y(t)-z(t)| &\le a(|x_0| + |z_0|)|x_0 - z_0|\\ &+ \int_{t_0}^{t} a(|y(s)-z(s)|)W(s,|y(s)-z(s)|,T|y(s)-z(s)|)ds.\end{aligned}$$

Thus we have

$$\begin{aligned} & m(t) - \int_{t_0}^{t} a(m(s))W(s, m(s), Tm(s))ds \\ & \leq a(|x_0| + |z_0|)m(t_0) \\ & < u(t) - \int_{t_0}^{t} a(u(s))W(s, u(s), Tu(s))ds \end{aligned}$$

if $a(|x_0| + |z_0|)m(t_0) < u(t_0)$. Hence $m(t) < u(t)$ for all $t \geq t_0$ by Lemma 2.1. Since (S-1) is stable we have

$$|u(t)| < c\delta, \quad t \geq t_0,$$

whenever $|u(t_0)| < \delta$. Further, we note that

$$|y_0 - z_0| \leq a(|x_0| + |z_0|)|y_0 - z_0| \leq \delta.$$

We have $|y(t) - z(t)| < c\delta, t \geq t_0$ whenever $|y_0 - z_0| < \delta$. Also it follows that if we have asymptotic stability then Equation (1) is asymptotically stable.

In following theorem we shall examine stability of perturbed equation by the nonlinear variation-of-constants formular, the comparison theorem and integral inequalities.

THEOREM 2.5. *Let the following conditions hold for perturbed equation (1):*

$$|g(t, y, Ty)| \leq W(t, |y|, T|y|),$$

where $W(t, 0, 0) = 0$ *and* $W(t, u, v)$ *is monotone nondecreasing with respect to* u *and* v *for each fixed* $t \geq 0, W(t, u, v) \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$

$$|\Phi(t, t_0, x_0)| \leq a(|x_0|), \quad |\Phi^{-1}(t, t_0, x_0)| \leq a(|x_0|), \quad t \geq t_0,$$

where $a \in C(\mathbb{R}^+)$ and $a(u)$ is nondecreasing in u. If the zero solution of the differential equation

$$\text{(S-2)} \qquad u' = a(u)W(t,u,Tu), \qquad u(t_0) = u_0, \quad a(|x_0|) \geq 1$$

is US(S, UAS), then the zero solution of (1) is also US(S, UAS).

PROOF. Let $y(t) = y(t, t_0, x_0)$ be a solution of (1) with an initial value $y(t_0, t_0, x_0) = x_0, t_0 \geq 0$. Then the solution of (1) is given by the formula

$$y(t) = \int_0^1 \Phi(t, t_0, x_0 s) ds x_0 + \int_{t_0}^t a(|y(s)|) W(s, |y(s)|, T|y(s)|) ds.$$

Next, let $u(t) = u(t, t_0, u_0)$ be the solution of (S-2) passing through (t_0, u_0) and let $a(|x_0|)|x_0| < u_0$. Then

$$|y(t)| - \int_{t_0}^t a(|y(s)|) W(s, |y(s)|, T|y(s)|) ds < u(t) - \int_{t_0}^t a(u) W(s, u, Tu) ds.$$

Therefore appling Lemma 2.1, we obtain that

$$|y(t)| < u(t), \qquad t \geq t_0.$$

Since the zero solution of (S-2) is US, for any $\epsilon > 0$ there exists a $\delta_1(\epsilon) > 0$ such that $|u_0| < \delta_1(\epsilon)$ implies $|y(t)| < \epsilon$ for all $t \geq t_0$. Set $\delta(\epsilon) = \frac{\delta_1(\epsilon)}{a(|x_0|)|x_0|}$. If $|x_0| < \delta(\epsilon)$, then take an $u_0 > 0$ so that $a(|x_0|)|x_0| < u_0 < \delta_1(\epsilon)$. Therefore we have $|y(t)| < \epsilon$ for all $t \geq t_0$, which completes the proof of the theorem.

COROLLARY 2.6. *If $f(t,x) = A(t)x$, where $A(t)$ is an $n \times n$ continuous matrix and the zero solution of (2) is US, that is, there exists a constant $K \geq 1$ such that*

$$|\Phi(t, t_0, x_0)\Phi^{-1}(s, t_0, x_0)| \leq K, \qquad t \geq t_0 \geq 0.$$

If the solution of the differential equation

$$\text{(S-3)} \qquad u' = KW(t,u,Tu), \quad u(t_0) = u_0 > K|x_0|$$

is US(S,UAS), then the zero solution of (1) is also US(S,UAS).

PROOF. We have

$$y(t,t_0,x_0) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,t_0)\Phi^{-1}(s,t_0)g(s,y(s),Ty(s))ds.$$

Then

$$|y(t)| - \int_{t_0}^{t} KW(s,|y(s)|,T|y(s)|)ds < u(t) - \int_{t_0}^{t} KW(s,u,Tu)ds.$$

Hence we obtain

$$|y(t)| < u(t), \quad t \geq t_0.$$

This completes the proof.

COROLLARY 2.7. *Let the following conditions hold for the differential equation (1) :*

$$|g(t,y,Ty)| \leq a(t)[|y| + \int_{t_0}^{t} b(s)|y|ds],$$

$$M(t_0) = 1 + K\int_{t_0}^{\infty} a(s)(\exp\int_{t_0}^{s} a(\tau) + b(\tau)d\tau)ds < \infty,$$

where $a, b \in C(\mathbb{R}^+)$ and $a, b \in L_1(\mathbb{R}^+)$. If the zero solution of (2) is US, then

$$|u(t)| \leq u_0(1 + K\int_{t_0}^{t} a(s)\exp\int_{t_0}^{s} a(\tau) + b(\tau)d\tau ds) \leq u_0 M(t_0).$$

Hence $u = 0$ of (S-3) is US and this implies that $y = 0$ of (1) is US by Corollary 2.6.

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