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A COLLOCATION METHOD FOR BIHARMONIC EQUATION

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ABSTRACT. An $O(h^4)$ cubic spline collocation method for biharmonic equation with a special boundary conditions is formulated and a fast direct method is proposed for the linear system arising when the cubic spline collocation method is employed. This method requires $O(N^2 \log N)$ arithmatic operations over an $N \times N$ uniform partition.

1. Introduction

We are interested in approximating the solution $u(x, y)$ of the biharmonic equation, which is the deflection of a transversely loaded simply supported plate,

$$\Delta^2 u = f \quad \text{in } \Omega = [a, b] \times [c, d] \tag{1}$$

with the boundary condtions

$$u = 0 \text{ and } \Delta u = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

where $\partial\Omega$ denotes the boundary of Ω. This problem is decomposable into the pair of problems

$$-\Delta v = -f \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \tag{3}$$

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$$(4) \qquad -\Delta u = -v \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Therefore we can find an approximation to the problem (1)-(2) by solving the second-order elliptic problem of the type (5)-(6) twice in a row.

In section 2, we first develop a cubic spline collocation method for the problem (5)-(6) where nodal point collocation with bicubic splines over an $N \times M$ uniform partition is applied to an $O(h^4)$ perturbation L' of the operator L. This idea was originated from E.N Houstis et.al [7]. In section 3, a fast direct method, which requires $O(NM \log NM)$ arithmatic operations, is proposed to solve the linear system arising from the cubic spline collocation method. The special structure of this linear system is much similar to the system of Bialecki et.al [2] who applied orthogonal spline collocation with piecewise Hermite bicubics to the problem (5)-(6). The fast direct method derived from the idea in [2] is based on the matrix decomposition and fast Fourier transform(FFT).

Many methods have been proposed for the numerical solution of the biharmonic equation. For example, Gustafsson [6], Braess and Peisker [3] and Monk [8] approached the problem by iterative finite element methods and Zhang [12] proposed a multilevel additive Schwarz method based on the finite element method with domain decomposition. Smith [10, 11], Ehrlich [5], Bauer and Reiss [1], Buzbee and Dorr [4] and Bjørstad [9] approached it by finite difference method based on the 13-point stencil. According to Bjørstad [9], the complexity of the above finite difference methods is between $O(N^{2.5} \log N)$ and $O(N^4)$. In [9], Bjørstad proposed a very efficient mixed method for a rectangular region, a fast direct method with an inner iterative part whose arithmatic complexity is $O(N^{7/3})$. But the error of the finite difference approximation based on the 13-point stencil is only $O(h^2)$ instead of $O(h^4)$ of the cubic spline collocation method in this paper. Even though the boundary conditions and the region is quite special, the new method in this paper is quite efficient and it may be generalized to a biharmonic equation with the boundary condition

$$u = h_1(x,y) \text{ and } \Delta u = h_2(x,y) \qquad \text{on } \partial\Omega.$$

2. Cubic spline collocation method

In this section we consider the cubic spline collocation method for the second-order elliptic partial differential equations

$$(5) \qquad Lw \equiv -(D_x^2 w + D_y^2 w) = g \qquad \text{in } \Omega$$

with the homogeneous boundary conditions

$$(6) \qquad w = 0 \quad \text{on } \partial\Omega.$$

Let $S_{\pi_x}^{(0)}$ be the one-dimensional cubic splines associated with the uniform partition $\pi_x = \{x_i = a + ih_x \,:\, 0 \le i \le N,\, h_x = (b-a)/N\}$, which vanishes at both ends $x = a$ and $x = b$. A basis $\{B_0, \cdots, B_N\}$ for $S_{\pi_x}^{(0)}$ can be defined by

$$\begin{aligned}
&B_0(x) = \hat{B}_0(x) - 4\hat{B}_{-1}(x), \quad B_1(x) = \hat{B}_1(x) - \hat{B}_{-1}(x),\\
&B_i(x) = \hat{B}_i(x), \quad i = 2, \cdots, N-2,\\
&B_{N-1}(x) = \hat{B}_{N-1}(x) - \hat{B}_{N+1}(x), \quad B_N(x) = \hat{B}_N(x) - 4\hat{B}_{N+1}(x),
\end{aligned}$$

where the basis function $\hat{B}_i$, for $-1 \le i \le n+1$, is the cubic spline with the support $[x_{i-2}, x_{i+2}]$ in one space variable over the extended uniform partition $\{x_i = a + ih_x \,:\, -3 \le i \le N+3,\, h_x = (b-a)/N\}$ such that

$$\begin{aligned}
&\hat{B}_i(x_{i\pm1}) = 1/6, && \hat{B}_i(x_i) = 2/3, && \hat{B}_i'(x_{i-1}) = 1/(2h_x)\\
&\hat{B}_i'(x_{i+1}) = -1/(2h_x), && \hat{B}_i''(x_{i\pm1}) = 1/h_x^2, && \hat{B}_i''(x_i) = -2/h_x^2.
\end{aligned}$$

In a similar manner we can define $S^{(0)}_{\pi_y}$ associated with the uniform partition $\pi_y = \{y_j = c + jh_y : 0 \le j \le M, h_y = (d-c)/M\}$, which vanishes at both ends $x = c$ and $x = d$. Then $\pi = \pi_x \times \pi_y$ is the uniform partition of Ω. Let $S^{(0)}_{\pi}$ be the two-dimensional bicubic splines with the partition π which satisfy the boundary conditions (6). The basis functions for $S^{(0)}_{\pi}$ can be constructed by forming the tensor product of basis functions of the one dimensional cubic splines $S^{(0)}_{\pi_x}$ and $S^{(0)}_{\pi_y}$. Throughout the notation $g_{i,j}$ denotes $g(x_i, y_j)$ for any function g.

In order to formulate the spline collocation method we need to define a high order perturbation L' of the operator L by

$$\begin{aligned} L'w_{i,j} = &-\frac{1}{12}[D_x^2 w_{i-1,j} + 10D_x^2 w_{i,j} + D_x^2 w_{i+1,j} + D_y^2 w_{i,j-1} \\ &+ 10D_y^2 w_{i,j} + D_y^2 w_{i,j+1}] \text{ at the interior points of } \Omega, \\ L'w_{i,j} = &\ Lw_{i,j} \text{ at the boundary points of } \Omega. \end{aligned}$$

The cubic spline collocation approximation $W \in S^{(0)}_{\pi}$ to the solution w of (5) and (6) is obtained by requiring that

1) The interior conditions

$$L'W_{i,j} = g_{i,j}, \quad i = 1, \cdots, N-1, \quad j = 1, \cdots, M-1, \tag{7}$$

2) The inner boundary conditions

$$\begin{cases} -D_x^2 W_{i,j} = g_{i,j}, & i = 0, N, \quad j = 1, \cdots, M-1, \\ -D_y^2 W_{i,j} = g_{i,j}, & j = 0, M, \quad i = 1, \cdots, N-1, \end{cases} \tag{8}$$

3) The corner conditions

$$-D_x^2 D_y^2 W_{i,j} = D_x^2 g_{i,j}, \quad i = 0, N, \ j = 0, M. \tag{9}$$

The uniqueness and existence of the cubic spline collocation approximation W is an immediate consequence of E.N Houstis et.al [7] and the optimal convergence of $O(h^4)$ follows from Theorem 1 which can be found in E.N Houstis et.al [7].

THEOREM 1. *Let W be the cubic spline collocation approximation of w. If $g \in C^2[\Omega]$, then the error bound is*

$$||D_x^k D_y^l (W-w)||_\infty \le \alpha_{k,l} h^{4-(k+l)},\ 0 \le k,l \le 3,\ 0 \le k+l \le 4,$$

where $h = \max\{h_x, h_y\}$ and $\alpha_{k,l}$ is independent of h.

To obtain the matrix forms of (7)-(9), let

$$W(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{M} C_{i,j} B_i(x) B_j(y).$$

Then the interior conditions becomes the system of linear equations

$$(\bar{A}_1 \otimes \bar{D}_2 + \bar{D}_1 \otimes \bar{A}_2)\vec{c} = \vec{g}, \tag{10}$$

where $\otimes$ denotes the tensor product of the matrices and

$$\bar{A}_1 = \left(a_{i,j}^{(1)}\right)_{i=1,j=0}^{N-1,M}, \quad a_{i,j}^{(1)} = -\frac{1}{12}\left[B_j''(x_{i-1}) + 10B_j''(x_i) + B_j''(x_{i+1})\right],$$
$$\bar{A}_2 = \left(a_{i,j}^{(2)}\right)_{i=1,j=0}^{M-1,N}, \quad a_{i,j}^{(2)} = -\frac{1}{12}\left[B_j''(y_{i-1}) + 10B_j''(y_i) + B_j''(y_{i+1})\right],$$
$$\bar{D}_1 = \left(d_{i,j}^{(1)}\right)_{i=1,j=0}^{N-1,M}, \quad d_{i,j}^{(1)} = B_j(x_i),$$
$$\bar{D}_2 = \left(d_{i,j}^{(2)}\right)_{i=1,j=0}^{M-1,N}, \quad d_{i,j}^{(2)} = B_j(y_i),$$
$$\vec{c} = (C_{0,0}, \cdots, C_{0,M}, C_{1,0}, \cdots, C_{1,M}, \cdots, C_{N,0}, \cdots, C_{N,M})^T,$$
$$\vec{g} = (g_{0,0}, \cdots, g_{0,M}, g_{1,0}, \cdots, g_{1,M}, \cdots, g_{N,0}, \cdots, g_{N,M})^T.$$

It follows from the corner conditions (9) that

$$C_{i,j} = -\frac{1}{36} h_x^2 h_y^2 D_x^2 g_{i,j}, \quad i = 0, N,\ j = 0, M. \tag{11}$$

Since Theorem 1 says the error $||W - w||_\infty$ is of $O(h^4)$, we may put $C_{i,j} = 0$, $i = 0, N$, $j = 0, M$ without losing any accuracy. Introducing $T_n(\alpha) = \text{tridiag}(1, \alpha, 1)$ of order n and

$$\begin{aligned}
\vec{G^i} &= (h_x^2 g_{i,1} - C_{i,0}, h_x^2 g_{i,2}, \cdots, h_x^2 g_{i,M-2}, \\
&\qquad\qquad h_x^2 g_{i,M-1} - C_{i,M})^T, \quad i = 0, N, \\
\vec{G_j} &= (h_y^2 g_{1,j} - C_{0,j}, h_y^2 g_{2,j}, \cdots, h_y^2 g_{N-2,j}, \\
&\qquad\qquad h_y^2 g_{N-1,j} - C_{N,j})^T, \quad j = 0, M,
\end{aligned}$$

the inner boundary condition (8) can be represented by

$$\begin{aligned}
T_{M-1}\vec{C^i} &= \vec{G^i}, \qquad i = 0, N, \\
T_{N-1}\vec{C_j} &= \vec{G_j}, \qquad j = 0, M,
\end{aligned}$$

where $T_{M-1} = T_{M-1}(4)$, $T_{N-1} = T_{N-1}(4)$ and

$$\begin{aligned}
\vec{C^i} &= (C_{i,1}, C_{i,2}, \cdots, C_{i,M-2}, C_{i,M-1})^T, \; i = 0, N, \\
\vec{C_j} &= (C_{1,j}, C_{2,j}, \cdots, C_{N-2,j}, C_{N-1,j})^T, \; j = 0, M.
\end{aligned} \tag{12}$$

Since $T_n(4)$ is nonsingular tridiagonal symmetric matrix, the inner boundary equations (8), and hence the boundary unknowns (12), can be solved at the cost of $O(N+M)$. Introducing the $n \times n$ pentadiagonal matrix

$$P_n(z,a) = \begin{pmatrix}
z-1 & a & 1 & 0 & & & & & & \\
a & z & a & 1 & 0 & & & & & \\
1 & a & z & a & 1 & 0 & & & & \\
0 & 1 & a & z & a & 1 & 0 & & & \\
 & & & \vdots & & & \vdots & & & \\
 & & & 1 & a & z & a & 1 & 0 & \\
 & & & 0 & 1 & a & z & a & 1 & \\
 & & & & 0 & 1 & a & z & a & \\
 & & & & & 0 & 1 & a & z-1 &
\end{pmatrix},$$

the matrix form (10) of the interior boundary conditions (7) after eliminating the corner and inner boundary unknowns (11) and (12) can be reformulated as

$$(h_y^2 A_1 \otimes D_2 + h_x^2 D_1 \otimes A_2)\vec{C} = \vec{G}, \tag{13}$$

where $A_1 = P_{N-1}(-18,8)$, $A_2 = P_{M-1}(-18,8)$, $D_1 = T_{N-1}(4)$, $D_2 = T_{M-1}(4)$,

$$\begin{aligned}
\vec{C} &= (C_{1,1},\cdots,C_{1,M-1},\ C_{2,1},\cdots,C_{2,M-1},\cdots,C_{N-1,1},\cdots, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad C_{N-1,M-1})^T, \\
\vec{G} &= (G_{1,1},\cdots,G_{1,M-1},\ G_{2,1},\cdots,G_{2,M-1},\cdots,G_{N-1,1},\cdots, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G_{N-1,M-1})^T,
\end{aligned}$$

and, for $i = 1,\cdots,N-1,\ j = 1,\cdots,M-1$,

$$\begin{aligned}
G_{i,j} = &-72h_x^2h_y^2\big[g_{i,j} - a_{i,0}^{(1)}\sum_{s=0}^{M} C_{0,s}\,d_{j,s}^{(2)} - d_{i,0}^{(1)}\sum_{s=0}^{M} C_{0,s}\,a_{j,s}^{(2)} \\
&- a_{i,N}^{(1)}\sum_{s=0}^{M} C_{N,s}\,d_{j,s}^{(2)} - d_{i,N}^{(1)}\sum_{s=0}^{M} C_{N,s}\,a_{j,s}^{(2)} - d_{j,0}^{(2)}\sum_{r=0}^{N} C_{r,0}\,a_{i,r}^{(1)} \\
&- a_{j,0}^{(2)}\sum_{r=0}^{N} C_{r,0}\,d_{i,r}^{(1)} - d_{j,M}^{(2)}\sum_{r=0}^{N} C_{r,M}\,a_{i,r}^{(1)} - a_{j,M}^{(2)}\sum_{r=0}^{N} C_{r,M}\,d_{i,r}^{(1)}\big].
\end{aligned} \tag{14}$$

Note that $\vec{G}$ can be calculated at the cost of $O(NM)$ instead of $O(NM^2+N^2M)$ since it follows from the definition of $a_{i,j}^{(1)}, a_{i,j}^{(2)}, b_{i,j}^{(1)}, b_{i,j}^{(2)}$ that the equation (14) becomes

$$G_{i,j} = -72h_x^2h_y^2\,g_{i,j}, \quad i = 3,\cdots,N-3,\ j = 3,\cdots,M-3,$$

and the rest of $G_{i,j}$'s can be obtained at the cost of $O(N+M)$. For example, the equation (14), for $i=1,\ j=1,\cdots,M-1$, becomes

$$\begin{aligned} G_{1,j} = & -72h_x^2h_y^2\big[g_{1,j} - a_{1,0}^{(1)} \sum_{s=j-1}^{j+1} C_{0,s}\, d_{j,s}^{(2)} - d_{1,0}^{(1)} \sum_{s=j-2}^{j+2} C_{0,s}\, a_{j,s}^{(2)} \\ & - d_{j,0}^{(2)} \sum_{r=0}^{3} C_{r,0}\, a_{1,r}^{(1)} - a_{j,0}^{(2)} \sum_{r=0}^{2} C_{r,0}\, d_{1,r}^{(1)} - d_{j,M}^{(2)} \sum_{r=0}^{3} C_{r,M}\, a_{1,r}^{(1)} \\ & - a_{j,M}^{(2)} \sum_{r=0}^{2} C_{r,M}\, d_{1,r}^{(1)}\big] \end{aligned}$$

where $C_{0,-1}=0$.

3. Algorithm

To solve the linear system (13), we formulate a fast direct algoritm based on the matrix decomposition which is highly suitable for parallel computations and employs the fast Fourier transform(FFT) to reduce the arithmatic operations involving matrix-vector multiplications.

Let $S_n=(s_{i,j})$ denote the unitary matrix of order n such that

$$s_{i,j} = \sqrt{\frac{2}{n+1}}\sin\frac{ij\pi}{n+1}, \quad i,j=1,\cdots,n,$$

and $diag(d_1,\cdots,d_n)$ the $n\times n$ diagonal matrix with diagonal elements d_k. The following Lemma is a well-known fact whose proof is omitted.

LEMMA 1. *For any real numbers z, a and α,*

$$S_nT_n(\alpha)S_n = diag(t_1,\cdots,t_n),$$
$$S_nP_n(z,a)S_n = diag(p_1,\cdots,p_n),$$

where

$$t_k = \alpha + 2\cos\frac{k\pi}{n+1}, \quad k=1,\cdots,n,$$
$$p_k = z-2-2a\cos\frac{k\pi}{n+1} + 4\cos^2\frac{k\pi}{n+1}, \quad k=1,\cdots,n.$$

Introducing, for $n = N-1, M-1$,

$$P_n = P_n(-18, 8), \quad T_n = T_n(4), \quad \Sigma_n = S_n T_n^{-1} S_n,$$

it follows from Lemma that

$$\Sigma_n S_n P_n S_n = \Lambda_n, \quad n = N-1, M-1, \tag{15}$$

where, for $n = N-1, M-1$,

$$\Lambda_n = diag(\lambda_1, \cdots, \lambda_n), \quad \lambda_k = \frac{\left(-2 + 2\cos\frac{k\pi}{n+1}\right)\left(10 + 2\cos\frac{k\pi}{n+1}\right)}{\left(4 + 2\cos\frac{k\pi}{n+1}\right)}.$$

Let I_n denote the identity matrix of order n. Using the facts (15)and the properties of the matrix tensor product, we see that

$$\begin{aligned}&(\Sigma_{N-1}S_{N-1} \otimes \Sigma_{M-1}S_{M-1})\left(h_y^2 A_1 \otimes D_2 + h_x^2 D_1 \otimes A_2\right)\\ &(S_{N-1} \otimes S_{M-1}) = h_y^2 \Lambda_{N-1} \otimes I_{M-1} + h_x^2 I_{N-1} \otimes \Lambda_{M-1},\end{aligned}$$

and hence that the equation (13) is equivalent to

$$\begin{aligned}&\left(h_y^2 \Lambda_{N-1} \otimes I_{M-1} + h_x^2 I_{N-1} \otimes \Lambda_{M-1}\right)(S_{N-1} \otimes S_{M-1})\vec{C}\\ &\qquad = (\Sigma_{N-1}S_{N-1} \otimes \Sigma_{M-1}S_{M-1})\vec{G},\end{aligned}$$

which shows that the following matrix decomposition algorithm for the equation (13) holds.

ALGORLTHM I

1. Compute $\vec{G^*} = (\Sigma_{N-1}S_{N-1} \otimes \Sigma_{M-1}S_{M-1})\vec{G}$.
2. $\left(h_y^2 \Lambda_{N-1} \otimes I_{M-1} + h_x^2 I_{N-1} \otimes \Lambda_{M-1}\right)\vec{C^*} = \vec{G^*}$.
3. Compute $\vec{C} = (S_{N-1} \otimes S_{M-1})\vec{C^*}$.

To consider the operations count and the parallelism of Algorithm I, we note that if we may regard any vector

$$\vec{Q} = (q_{1,1}, \cdots, q_{1,m}, q_{2,1}, \cdots, q_{2,m}, \cdots, q_{n,1}, \cdots, q_{n,m})^T$$

as an $m \times n$ matrix Q whose j-th column is $(q_{j,1}, \cdots, q_{j,m})^T$, then it follows from the properties of the tensor product that we may view step 1 and step 3 as matrix forms

$$G^* = \Sigma_{M-1} S_{M-1} G S_{N-1} \Sigma_{N-1},$$
$$C = S_{M-1} C^* S_{N-1}.$$

In this manner all steps are highly suitable for parallel computaions since each column of $S_{M-1}GS_{N-1}$ and $S_{M-1}C^*S_{N-1}$ can be calculated separately using fast Fourier transform routines. Noting that the matrices Σ_{N-1} and Σ_{M-1} are diagonal, it is easy to see that the multiplication involving the matrices Σ_{N-1} and Σ_{M-1} in step 1 can be performed within $2(N-1)(M-1)$ arithmatic operations. If FFT is employed to perform the multiplication by S_{N-1} and S_{M-1} in step 1 and step 3, each of these steps can be performed at the cost of $O(NM \log NM)$. Step 2 requires only $NM - 1$ arithmatic operations since it is a diagonal system of order $(N-1)(M-1)$. Therefore the total cost of Algorithm I is $O(NM \log NM)$, which is almost optimal considering the number of unknowns is $(N-1)(M-1)$.

Finally the biharmonic problem (1)-(2) can be solved by the following Algorithm II.

ALGORLTHM II

Repeat the following with $g = -f$ and then with $g = -W$.

1. Compute $C_{i,j} = -\frac{1}{36} h_x^2 h_y^2 D_x^2\, g_{i,j}, \quad i = 0, N, j = 0, M.$

2. Compute $\vec{G}^i$, $i = 0, N$, and $\vec{G}_j$, $j = 0, N$, and solve

$$T_{M-1}\vec{C}^i = \vec{G}^i,\ i = 0, N, \quad T_{N-1}\vec{C}_j = \vec{G}_j,\ j = 0, M.$$

3. Compute $\vec{G}$.
4. Apply Algorithm I to solve the equation (13).

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