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# BARRELED FUZZY LINEAR SPACES

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ABSTRACT. We provide a definition of a barreled fuzzy linear space and prove that the quotient and product of barreled fuzzy linear spaces are barreled.

## 1. Introduction

The concepts of a fuzzy vector space and a fuzzy topological vector space were introduced in [6]. These ideas were modified by Katsaras in [2] and in [3] Katsaras defined a fuzzy norm on a vector space. In [4, 5] the author studied some of the properties of those fuzzy topological vector spaces whose fuzzy topology is given by some fuzzy neighborhood system. Krishna and Sarma[7] studied the topological generation and normability of fuzzy topological vector spaces by observing the equivalence of fuzzy topology on a vector space, obtained in several different ways. Bakier and El-Saady[1] applied the concept of a fuzzy set to the elementary theory of order and vector spaces and studied some of the properties of fuzzy topological ordered vector spaces.

In the present paper, We provide a definition of a barreled fuzzy linear space and prove that the quotient and product of barreled fuzzy linear spaces are barreled.

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## 2. Preliminaries

Let $X$ be a non-empty set and $I$ the unit interval. A fuzzy set in $X$ is an element of the set $I^X$ of all functions from $X$ to $I$. A fuzzy topology on a set $X$ is subfamily $\Im$ of $I^X$ satisfying the following conditions:

(1) $\Im$ contains every constant fuzzy set in $X$.

(2) If $\mu, \rho \in \Im$, then $\mu \wedge \rho \in \Im$.

(3) If $\mu_\alpha \in \Im$ for each $\alpha \in J$, then $\sup_{\alpha \in J} \mu_\alpha \in \Im$.

A fuzzy topological space is a set on which there is given a fuzzy topology $\Im$. The elements of $\Im$ are called open fuzzy sets in $X$. A fuzzy set $\mu \in I^X$ is called closed iff $\mu^c$ is open.

Let $X$ be a fuzzy topological space and $x \in X$. A fuzzy set $\mu$ in $X$ is called a neighborhood of $x$ if there exists an open fuzzy set $\rho$ with $\rho \leq \mu$ and $\rho(x) = \mu(x) > 0$.

A map $f$, from a fuzzy topological space $X$ to a fuzzy topological space $Y$, is called continuous if $f^{-1}(\mu)$ is open in $X$ for each open fuzzy set $\mu$ in $Y$. It is equivalent that $f^{-1}(\rho)$ is closed in $X$ for each closed fuzzy set $\rho$ in $Y$.

Suppose that $E$ is a vector space over $K$, where $K$ is the space of either the real or the complex numbers. If $\mu, \rho$ are fuzzy sets in $E$, then the fuzzy set $\mu \oplus \rho$ is defined by

$$\mu \oplus \rho(x) = sup\{\mu(x_1) \wedge \rho(x_2) \mid x = x_1 + x_2\}.$$

For $x \in E$ and $\mu$ a fuzzy set in $E$, $x \oplus \mu$ is defined by

$$x \oplus \mu(y) = \mu(y - x).$$

A fuzzy set $\mu$ in $E$ is called :

(1) convex if $t\mu \oplus (1-t)\mu \leq \mu$ for all $0 \leq t \leq 1$

(2) balanced if $t\mu \leq \mu$ for $|t| \leq 1$

(3) absorbing if $\sup_{0<t<1} t\mu = 1$

(4) a fuzzy subspace if $\mu \oplus \mu \leq \mu$ and $t\mu \leq \mu$ for every scalar $t$.

Let $\mu_i$, $i = 1, 2, \ldots, n$, be a fuzzy set in $E_i$, then $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ is the fuzzy set $\mu$ in $E = \prod_{i=1}^{n} E_i$ defined by

$$\mu(x_1, x_2, \ldots, x_n) = inf\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\}.$$

A fuzzy linear topology on a vector space $E$ over $K$ is a fuzzy topology $\Im$ such that the two mappings

$$+ : E \times E \to E, \quad (x, y) \mapsto x + y$$

$$\cdot \; : K \times E \to E, \quad (t, x) \mapsto tx$$

are continuous where $K$ is equipped with the fuzzy topology generated by usual topology of $K$ and $K \times E$, $E \times E$ have the corresponding product fuzzy topologies. A vector space $E$, with a fuzzy linear topology, is called a fuzzy topological vector space or a fuzzy linear space.

## 3. Main Part

Now we prove the main results.

DEFINITION 1. A fuzzy set $\mu$ in fuzzy linear space $E$ is called a *barrel* if $\mu$ is closed, convex, balanced and absorbing.

DEFINITION 2. A fuzzy linear space $E$ is said to be *barreled* if every barrel fuzzy set in $E$ is a neighborhood of 0 in $E$.

LEMMA 1. *Let $E$, $F$ be vector spaces over $K$ and let $f : E \to F$ be a linear map. If $\rho$ is a convex(balanced, absorbing) fuzzy set in $F$, then $f^{-1}(\rho)$ is a convex(balanced, absorbing) fuzzy set in $E$.*

Let $\Im$ be a fuzzy linear topology on $E$ and $\mu$ a fuzzy subspace. The quotient fuzzy topology on $E/\mu$ is the finest of all fuzzy topologies on

$E/\mu$ for which the quotient mapping $q : E \to E/\mu$, $q(x) = x \oplus \mu$, is continuous. The quotient fuzzy topology on $E/\mu$ consists of all fuzzy sets $\rho$ in $E/\mu$ for which $q^{-1}(\rho) \in \Im$ and is linear. Also the quotient map $q$ is linear and open.

THEOREM 1. *Let a fuzzy linear space $E$ be barreled, and $\mu$ a fuzzy subspace of $E$. Then the quotient space $E/\mu$ is barreled.*

PROOF. Let $\rho$ be a barrel in $E/\mu$, and $q : E \to E/\mu$ a quotient map. Then $q^{-1}(\rho)$ is a barrel, hence $q^{-1}(\rho)$ is a neighborhood of 0 in $E$. Also the image under $q$ of a neighborhood of 0 in $E$ is a neighborhood of 0 in $E/\mu$[2], and $\rho = q(q^{-1}(\rho))$. Thus $\rho$ is a neighborhood of 0 in $E/\mu$.

Let $\{E_\alpha\}$ be a family of fuzzy linear spaces over $K$ and let $E = \prod_{\alpha \in J} E_\alpha$ with the product fuzzy topology. Then $E$ is a fuzzy linear space.

THEOREM 2. *Let $\{E_\alpha\}_{\alpha \in J}$ be a finite family of barreled fuzzy linear space over $K$ and let $E = \prod_{\alpha \in J} E_\alpha$ with the product fuzzy topology. Then $E$ is barreled.*

PROOF. Let $\pi_\alpha : E \to E_\alpha$ denote the $\alpha$-th projection map, and $\rho$ a barrel fuzzy set in $E$. $\rho_\alpha \in I^{E_\alpha}$ and $x = (x_\alpha)_{\alpha \in J} \in E$. Then there exist $\alpha_0 \in J$ with $\rho(x) = \rho_{\alpha_0} \circ \pi_{\alpha_0} = \rho_{\alpha_0}(x_{\alpha_0})$. So $\rho_{\alpha_0}$ is a barrel in $E_{\alpha_0}$ and neighborhood of 0 in $E_{\alpha_0}$. Hence there is a $\mu_{\alpha_0} \in I^{E_{\alpha_0}}$ such that $\mu_{\alpha_0}(x_{\alpha_0}) \leq \rho_{\alpha_0}(x_{\alpha_0})$ and $\mu_{\alpha_0}(0) = \rho_{\alpha_0}(0) > 0$. Let $\mu(x) = min\{\mu_\alpha(x_\alpha)\}_{\alpha \in J}$. Then $\mu(0, 0, \dots, 0) = \rho(0, 0, \dots, 0) > 0$ and $\mu(x) = min\{\mu_\alpha(x_\alpha)\}_{\alpha \in J} \leq \rho_{\alpha_0}(x_{\alpha_0}) = \rho(x)$. Thus $\rho$ is a neighborhood of zero in $E$.

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