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ON THE FAMILIES OF PLANE QUARTICS DEGENERATING TO A PLANE QUARTIC WITH AN ORDINARY CUSP OF MULTIPLICITY THREE

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There are up to projective equivalence two quartics with an ordinary cusp of multiplicity three[5]. Their equations are $x^4 - y^3z = 0$ and $x^4 - x^3y + y^3z = 0$.

We say that $p : X \to \Delta$, Δ the unit disk of $\mathbb{C}$, is *a right family of plane quartics degenerating to C* if all the fibers over $t \neq 0$ are nonsingular plane quartics and generically projectively independent with $X_0 = C$. This family always gives a morphism ϕ from $\Delta^* = \Delta - \{0\}$ to a punctured arc in $\mathcal{M}_3$ the moduli space of all isomorphism classes of all genus three smooth projective algebraic curves which is defined by $\phi(t) = [X_t]$ the isomorphism classes of X_t. In this paper, we compute a stable curve of genus three whose isomorphism class is the limit point $\lim_{t\to 0}[X_t]$ in $\overline{\mathcal{M}}_3$ and show that it lies in $\mathcal{M}_3$ except the case that the total surface X has a triple point. We call this limit point *the stable model of C obtained from X*. The proof can be divided into two parts. One part is to find the desingularization $\tilde{X}$ of X with the new fiber over $t = 0$ a connected divisor with normal crossings. The other part is to replace the central fiber with a semistable curve (the semistable reduction theorem[1, 4]). A *semistable curve* is a nodal curve without smooth rational components of self intersection number (-1). Contracting the rational components of self intersection number

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(-2), we get a *stable curve.* In this paper, the proof of the second part will be shortly sketched since the similar proof have been done in [2] and [3] by the same author.

We fix that C is a plane quartic with an ordinary cusp of multiplicity three given by either equation $f(x,y,z)$ in the above, $P=(0:0:1)$ the singular point of C and X the right family of plane quartics degenerating to C. Then the equation of X is given by

$$F(x,y,z)=f(x,y,z)+tg_1+t^2g_2+\cdots+t^ng_n$$

where $g_i = g_i(x,y,z)$ are homogeneous functions of degree 4 and $g_1 \neq 0$. Since X is a right family, $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ cannot simultaneously be zero. Therefore the singular points of X can occur along $t=0$. From the following equations

$$\frac{\partial F}{\partial x}=\frac{\partial f}{\partial x}+t(\sum_{i=1}^{n}\frac{\partial g_i}{\partial x}),\ \frac{\partial F}{\partial y}=\frac{\partial f}{\partial y}+t(\sum_{i=1}^{n}\frac{\partial g_i}{\partial y}),$$
$$\frac{\partial F}{\partial z}=\frac{\partial f}{\partial z}+t(\sum_{i=1}^{n}\frac{\partial g_i}{\partial z}),\frac{\partial F}{\partial t}=g_1+2tg_2+3t^2g_3+\cdots+nt^{n-1}g_n,$$

X is nonsingular if and only if the curve $g_1=0$ does not pass the singular point of C, which in the present case is equivalent to $g_1(P)\neq 0$. If X is nonsingular, the stable model of C obtained from X is a smooth curve from the usual stable reduction [2].

Now assume that X is singular, i.e., $g_1(P)=0$. We remark that X can be considered a right family unless that P is a singular point of the curve defined by g_i for all i. Since X has only one singular point $(0:0:1)$ with $t=0$, we can work on the neighborhood N of $z=1$. We also work with $f(x,y,z)=y^3z-x^4$ since the term y^3z does not affect any other things except the total transform of C. We write $g_i(x,y,1)$ as $g_i(x,y)$.

Let X be the family given by

$$y^3 - x^4 + \sum_{i=1}^{n} t^i g_i(x,y) = 0.$$

It has the only singular point at origin $P = (0,0,0) \in \mathbb{A}^3_{(x,y,t)}$. From our choice as mentioned above, (0,0) is not a multiple point of $g_i = 0$ for some i.

The case that P is a double point of X. Then X can be given by

$$F(x,y,t) = y^3 - x^4 + t(b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + [3]) \\ + t^2(c_{00} + c_{10}x + c_{01}y + [2]) + t^3(\cdots)$$

where at least one of b_{10}, b_{01} and c_{00} is not zero. Let $\pi_1 : \tilde{\mathbb{A}}^3 \to \mathbb{A}^3$ be the blow-up of $\mathbb{A}^3$ at the origin and $\tilde{X}$ the proper transform of X under π_1. Let

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{t}{t_1}$$

be the equation of $\tilde{\mathbb{A}}^3$ in $\mathbb{A}^3{}_{(x,y,t)} \times \mathbb{P}^2{}_{(x_1:y_1:t_1)}$. We may take as local coordinates on the open set U_1 of $x_1 \neq 0$ the functions x, $y_1 = y/x$ and $t_1 = t/x$; on V_1 of $y_1 \neq 0$ the functions $x_1 = x/y$, y and $t_1 = t/y$; on the open set W_1 of $t_1 \neq 0$ the functions $x_1 = x/t$, $y_1 = y/t$ and t. We let $\pi_1 = \pi_1|_{\tilde{X}} : \tilde{X} \to X$, $f_1 = p \circ \pi_1 : \tilde{X} \to \Delta$.

Then the defining functions of the proper transform $\tilde{X}$ of X and of the central fiber over $t = 0$ on each open set are as follows respectively.

$$\tilde{X} \cap U_1 : y_1^3 x - x^2 + t_1(b_{10} + b_{01}y_1 + b_{20}x + b_{11}xy_1 + b_{02}xy_1^2 + [x^2]) \\ + t_1^2(c_{00} + c_{10}x + c_{01}xy_1 + [x^2]) + t_1^3 x(\cdots);$$

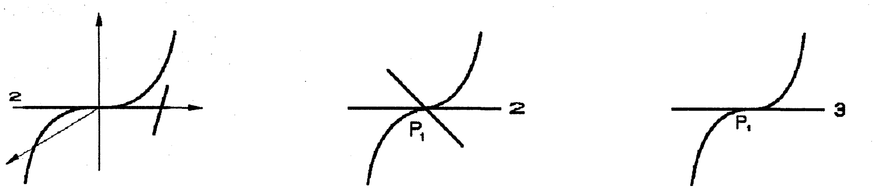
$$\tilde{X} \cap V_1 : y - x_1^4 y^2 + t_1(b_{10}x_1 + b_{01} + b_{20}x_1^2 y + b_{11}x_1 y + b_{02}y + [y^2]) \\ + t_1{}^2(c_{00} + c_{10}x_1 y + c_{01}y + [y^2]) + t_1{}^3 y(\cdots);$$

$$\tilde{X} \cap W_1 : y_1^3 t - x_1^4 t^2 + (b_{10}x_1 + b_{01}y_1 + b_{20}x_1^2 t + b_{11}x_1 y_1 t + b_{02}y_1^2 t \\ + [t^2]) + (c_{00} + c_{10}x_1 t + c_{01}y_1 t + [t^2]) + t(\cdots),$$

and

$$
\begin{aligned}
U_1 \cap f_1^*(0) &= (t) = (t_1) + (x) \\
&= (t_1,\ y_1^3 x - x^2) + (x,\ b_{10}t_1 + b_{01}y_1t_1 + c_{00}t_2) \\
&= (t_1,\ y_1^3 - x) + 2(x,\ t_1) + (x,\ b_{10} + b_{01}y_1 + c_{00}t_1) \\
&= C_1 + 2E_1 + F_1; \\
V_1 \cap f_1^*(0) &= (t) = (t_1) + (y) \\
&= (t_1,\ y - x_1^4 y^2) + (y,\ b_{10}t_1x_1 + b_{01}t_1 + c_{00}t_1^2) \\
&= (t_1,\ 1 - x_1^4 y) + 2(y,\ t_1) + (y,\ b_{10}x_1 + b_{01} + c_{00}t_1) \\
&= C_1 + 2E_1 + F_1; \\
W_1 \cap f_1^*(0) &= (t) = (t,\ b_{10}x_1 + b_{01}y_1 + c_{00}) = F_1.
\end{aligned}
$$

In the above, (g) means the zero locus of a function g. The following *figure 1* is the central fiber of $f_1 : \tilde{X} \to \Delta$.



$b_{10} \neq 0$ $\qquad$ $b_{10} = 0,\ b_{01} \neq 0$ $\qquad$ $b_{10} = b_{01} = 0$

figure 1

If $b_{10} \neq 0$, then $\tilde{X}$ is smooth and the usual stable reduction process, that is in this case two consecutive blow ups which makes the fiber over $t = 0$ one with normal crossings, the base change of the total order 18, the desingularizations of the total surface and contractions of rational components of self intersection number (-1) and (-2) (see *figure 2*), gives a smooth curve (more precisely trigonal curve totally ramified at 5 points). If one want to know the details of the proof, one can do it as we have done in [2] or [3].

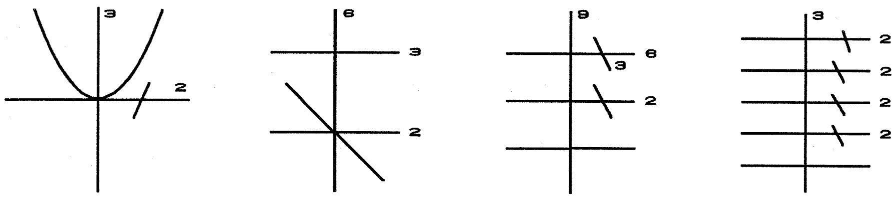


figure 2

If $b_{10} = 0$, then $\tilde{X}$ has only one singular point P_1 which is a double point on $\tilde{X} \cap U_1$. We now desingularize $\tilde{X}$. For the sake of equations, we in fact work on $N_1 = \tilde{X} \cap U_1 \subset \mathbb{A}^3{}_{(x,y_1,t_1)}$ without making any distinction from $\tilde{X}$. Take a blow up $\pi_2 : \tilde{\mathbb{A}}^3 \to \mathbb{A}^3$ and write $\tilde{X}^{(2)} = \tilde{N}_1$ the proper transform of N_1 under π_2 and $\pi_2 = \pi_2|_{\tilde{X}^{(2)}} : \tilde{X}^{(2)} \to \tilde{X}$. To describe $\tilde{X}^{(2)}$, let

$$\tilde{\mathbb{A}}^3 = \left\{ \frac{x}{x_2} = \frac{y_1}{y_2} = \frac{t_1}{t_2} \right\} \subset \mathbb{A}^3{}_{(x,y_1,t_1)} \times \mathbb{P}^2{}_{(x_2:y_2:t_2)}$$

and take as local coordinates on the open set U_2 of $x_2 \neq 0$ the functions x, $y_2 = y_1/x$ and $t_2 = t_1/x$; on V_2 of $y_2 \neq 0$ $x_2 = x/y_1$, y_1 and $t_2 = t_1/y_1$; on W_2 of $t_2 \neq 0$ $x_2 = x/t_1$, $y_2 = y_1/t_1$ and t_1. We let $f_2 = f_1 \circ \pi_2 : \tilde{X}^{(2)} \to \Delta$.

On each neighborhood U_2, V_2 and W_2 the defining function of the proper transform $\tilde{X}^{(2)}$ of $\tilde{X}$ is the following.

$$\begin{aligned}
\tilde{X}^{(2)} \cap U_2 &: y_2^3x^2 - 1 + t_2(b_{01}y_2 + b_{20} + b_{11}xy_2 + b_{02}x^2y_2^2 + [x]) \\
&\quad + t_2^2(c_{00} + c_{10}x + c_{01}x^2y_2 + [x^2]) + t_2^3x^2(\ \cdots\ ); \\
\tilde{X}^{(2)} \cap V_2 &: x_2y_1^2 - x_2^2 + t_2(b_{01} + b_{20}x_2 + b_{11}x_2y_1 + b_{02}x_2y_1^2 + [x_2^2y_1]) \\
&\quad + t_2^2(c_{00} + c_{10}x_2y_1 + c_{01}x_2y_1^2 + [x_2^2y_1^2]) + t_2^3x_2y_1^2(\ \cdots\ ); \\
\tilde{X}^{(2)} \cap W_2 &: x_2y_2^3{t_1}^2 - x_2^2 + (b_{01}y_2 + b_{20}x_2 + b_{11}x_2y_2t_1 + b_{02}x_2y_2^2t_1^2 \\
&\quad + [x_2^2t_1]) + (c_{00} + c_{10}x_2t_1 + c_{01}x_2y_2t_1^2 + [x_2^2t_1^2]) + t_1^2x_2(\ \cdots\ ).
\end{aligned}$$

Then we have

$$\begin{aligned}
U_2 \cap f_2^*(0) &= (t_1) + (x) = (t_2) + 2(x) \\
&= (t_2,\ y_2^3x^2 - 1) + 2(x,\ -1 + b_{01}y_2t_2 + b_{20}t_2 + c_{00}t_2^2); \\
V_2 \cap f_2^*(0) &= (t_1) + (x) = (t_2) + 2(y_1) + (x_2) \\
&= (t_2,\ x_2y_1^2 - x_2^2) + 2(y_1,\ -x_2^2 + b_{01}t_2 + b_{20}x_2t_2 + c_{00}t_2^2) \\
&\quad + (x_2,\ b_{01}t_2 + c_{00}t_2^2) \\
&= (t_2,\ y_1^2 - x_2) + 2(x_2,\ t_2) + (x_2,\ b_{01} + c_{00}t_2) \\
&\quad + 2(y_1,\ -x_2^2 + b_{01}t_2 + b_{20}x_2t_2 + c_{00}t_1^2); \\
W_3 \cap f_2^*(0) &= (t_1) + (x) = 2(t_1) + (x_2) \\
&= (x_2,\ b_{01}y_2 + c_{00}) + 2(t_1,\ -x_2^2 + b_{01}y_2 + b_{20}x_2 + c_{00}).
\end{aligned}$$

If $b_{10} = 0$ and $b_{01} \neq 0$, then $\tilde{X}^{(2)}$ is smooth and the *figure 3*(a) is the new fiber over $t = 0$.

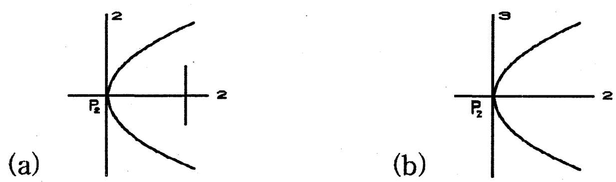


figure 3

By the consecutive blow ups of the total surface $\tilde{X}^{(2)}$ at P_2, the base changes of the total order 40, the singularizations(*figure 4*) and the contractions of some rational components, we get a smooth curve of genus three which is 4-gonal over $\mathbb{P}^1$ totally ramified at four points.

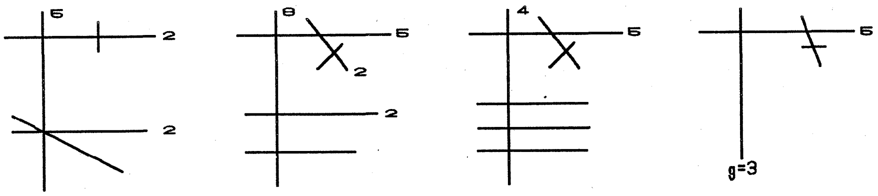


figure 4

If $b_{10} = b_{01} = 0$, and $c_{00} \neq 0$, then $\tilde{X}^{(2)}$ has a double point P_2 in the open set $N_2 = V_2 \cap \tilde{X}^{(2)} \subset V_2 \cong \mathbb{A}^3_{(x_2, y_1, t_2)}$ (*Figure 3*(b)). Therefore we need more blow-ups. As before, we define $\tilde{V}_2$, $\tilde{N}_2 = \tilde{X}^{(3)}$ and $\pi_3 : \tilde{X}^{(3)} \to \tilde{X}^{(2)}$. Since

$$\tilde{V}_2 = \left\{ \frac{x_2}{x_3} = \frac{y_1}{y_3} = \frac{t_2}{t_3} \right\} \subset \mathbb{A}^3_{(x_2, y_1, t_2)} \times \mathbb{P}^2_{(x_3 : y_3 : t_3)},$$

we take as local coordinates on the open set $U_3 = \{x_3 \neq 0\}$ the functions x_2, $y_3 = y_2/x_3$ and $t_3 = t_2/x_3$; on $V_3 = \{y_3 \neq 0\}$ the functions $x_3 = x_2/y_1$, y_1 and $t_3 = t_2/y_1$; on $W_3 = \{t_3 \neq 0\}$ the functions $x_3 = x_2/t_2$, $y_3 = y_1/t_2$ and t_2.

On each open neighborhood U_3, V_3, W_3, $\tilde{X}^{(3)}$ is given by the following equations respectively:

$$x_2 y_3^2 - 1 + t_3(b_{20} + b_{11} x_2 y_3 + b_{02} x_2^2 y_3^2 + [x_2^2 y_3])$$
$$+ t_3^2(c_{00} + c_{10} x_2^2 y_3 + c_{01} x_2^3 y_3 + [x_2^2 y_3^2]) + x_2^4 y_3^2 t_3^3(\cdots);$$
$$x_3 y_1 - x_3^2 + t_3(b_{20} x_3 + b_{11} x_3 y_1 + b_{02} x_3 y_1^2 + [x_3^2 y_1^2])$$
$$+ t_3^2(c_{00} + c_{10} x_3 y_1^2 + c_{01} x_3 y_1^3 + [x_3^2 y_1^4]) + x_3 y_1^4 t_3^3(\cdots);$$
$$x_3 y_3^2 t_2 - x_3^2 + (b_{20} x_3 + b_{11} x_3 y_3 t_2 + b_{02} x_3 y_3^2 t_2^2 + [x_3^2 y_3^2 t_2^2])$$
$$+ (c_{00} + c_{10} x_3 y_3 t_2^2 + c_{01} x_3 y_3^2 t_2^3 + [x_3^2 y_3^2 t_2^4]) + x_3 y_3^2 t_2^4(\cdots).$$

But $\tilde{X}^{(3)}$ is not smooth either. Its singular point lies in $\tilde{X}^{(3)} \cap V_3$. As we have done, we define N_3, π_4, U_4, V_4, W_4, $\tilde{X}^{(4)}$ and f_4 in the same way. Then $\tilde{X}^{(4)}$ is locally given by as follows on U_4, V_4 and W_4 respectively.

$$\begin{aligned}
&y_4 - 1 + t_4(b_{20} + b_{11}x_3y_4 + b_{02}x_3^2y_4^2 + [x_3y_4^2]) \\
&\qquad + t_4^2(c_{00} + c_{10}x_3^4y_4^2 + c_{01}x_3^4y_4^3 + [x_3^6y_4^4]) + x_3^6y_4^4t_4^3(\cdots); \\
&x_4 - x_4^2 + t_4(b_{20}x_4 + b_{11}x_4y_1 + b_{02}x_4y_1^2 + [x_4y_1^3]) \\
&\qquad + t_4^2(c_{00} + c_{10}x_4y_1^3 + c_{01}x_4y_1^4 + [x_4^2y_1^6]) + x_4y_1^6t_4^3(\cdots); \\
&x_4y_4 - x_4^2 + (b_{20}x_4 + b_{11}x_4y_4t_3 + b_{02}x_4y_4^2t_3^2 + [x_4y_4^2t_3^2]) \\
&\qquad + (c_{00} + c_{10}x_4y_4^2t_3^3 + c_{01}x_4y_4^3t_3^4 + [x_4^2y_4^4t_3^6]) + x_4y_4^4t_3^6(\cdots).
\end{aligned}$$

Now $\tilde{X}^{(4)}$ being smooth, we compute the central fiber $f_4^*(0)$.

$$\begin{aligned}
f_4^*(0) \cap U_4 &= (t_2) + 2(y_1) + (x_2) \\
&= (t_3) + 4(y_1) + (x_3) = (t_4) + 4(y_4) + 6(x_3) \\
&= (t_4,\ y_4 - 1) + 4(y_4,\ -1 + b_{20}t_4 + c_{00}t_4^2) \\
&\quad + 6(x_3,\ y_4 - 1 + b_{20}t_4 + c_{00}t_4^2); \\
f_4^*(0) \cap V_4 &= (t_3) + 4(y_1) + (x_3) = (t_4) + 6(y_1) + (x_4) \\
&= (t_4, x_4 - x_4^2) + 6(y_1, x_4 - x_4^2 + b_{20}x_4t_4 + c_{00}t_4^2) \\
&\quad + (x_4,\ c_{00}t_4^2) \\
&= 3(t_4,\ x_4) + (t_4,\ 1 - x_4) \\
&\quad + 6(y_1,\ x_4 - x_4^2 + b_{20}x_4t_4 + c_{00}t_4^2); \\
f_4^*(0) \cap W_4 &= 6(t_3) + 4(y_4) + (x_4) \\
&= 6(t_3,\ x_4y_4 - x_4^2 + b_{20}x_4 + c_{00}) \\
&\quad + 4(y_4,\ -x_4^2 + b_{20}x_4 + c_{00}).
\end{aligned}$$

Therefore it consists of the 6-tuple exceptional curve, two quadruple lines (possibly not distinct), a triple one and the forth proper transform of the original central curve C which is a smooth rational curve (*figure 5*). In *figure 5* parallelogram represents $\mathbb{P}^2_{(x_4:y_4:t_4)}$. Now the total base change of order 12 followed by desingularizations and contractions will give us a smooth curve of genus three that is a totally ramified trigonal curve.

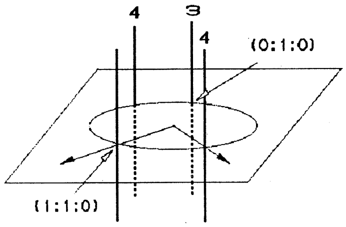


figure 5

The remaining case is that X can have a triple point. Therefore we have proved the following.

THEOREM. *Let C be a plane quartic with an ordinary cusp of multiplicity three and $p : X \to \Delta$ a right family of plane quartics degenerating to C. Then the stable model of C obtained from X is a smooth curve except that the total surface X has triple point along* $t = 0$.

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