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THE EXISTENCE THEOREM OF THE GENERALIZED ANALYTIC OPERATOR–VALUED FEYNMAN INTEGRAL

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ABSTRACT. We will investigate conditions of smooth measure μ for which analytic operator-valued Feynman integral associated with μ exist.

1. Introduction

We consider a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X,m)$ where X is locally compact separable metric space and m is a positive Radon measure on X with $Supp[m] = X$. S is the family of all smooth measures on X. Let $M = (\Omega, X_t, \zeta, P_x)$ be a Hunt process on X which is m-symmetric and associated with $(\mathcal{E},\mathcal{F})$. For a given smooth measure μ, we denote by A^μ the unique positive continuous additive functional such that μ is the Revuz measure of A^μ. Let $\mu = \mu_+ - \mu_-$ be a signed Borel measure on X. If μ_+ and μ_- are smooth measures, then we write $\mu \in S - S$. For a Borel measure v on X, $L^2(X,v)$ is sometimes written $L^2(v)$ when the underlying context is clear.

For $\mu \in S - S$, we put

$$\mathcal{E}_\mu(f,g) = \mathcal{E}(f,g) + \int_X f(x)g(x)\mu(dx)$$

for all $f,g \in \mathcal{F} \cap L^2(|\mu| + m)(= \mathcal{F}^\mu)$. It can be shown that $\mathcal{F}^\mu$ is dense in $L^2(m)$.

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We consider the case where X is the Euclidean space $\mathbb{R}^d$ and m is a Lebesgue measure on $\mathbb{R}^d$.

The Dirichlet forms are defined on subspaces of $L^2(\mathbb{R}^d)$ (over the scalar field $\mathbb{R}$), the space of $\mathbb{R}$ -valued, Borel measurable functions on $\mathbb{R}^d$ which are square integrable with respect to Lebesgue measure m. However, it is essential to the Feynman integral that the functions be from the space $L^2(\mathbb{R}^d, \mathbb{C})$ (*over* $\mathbb{C}$) of square-integrable, complex-valued functions.

Given $\omega \in \Omega = C([0,\infty), \mathbb{R}^d)$ and $\mu \in S - S$, let $F^\mu(\omega) = e^{-A_t^\mu(\omega)}$. Given $t > 0$, $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$ and $x \in \mathbb{R}^d$, consider the expression

(A)
$$(J^t(F^\mu)\psi)(x) = E_x\{e^{-A_t^\mu(\omega)}\psi(\omega(t))\} = \int_{\Omega_x} e^{-A_t^\mu(\omega)}\psi(\omega(t))\, dP_x(\omega)$$

where Ω_x is the set of $\omega \in C([0,\infty), \mathbb{R}^d)$ such that $\omega(0) = x$ and P_x is the probability measure associated with the Brownian paths in $\mathbb{R}^d$ which start at x at time 0. The operator-valued function space integral $J^t(F^\mu)$ exists for $t > 0$ if (A) defines $J^t(F^\mu)$ as an element of $\mathcal{L}(L^2(\mathbb{R}^d, \mathbb{C}))$, the space of bounded linear operators on $L^2(\mathbb{R}^d, \mathbb{C})$. If $J^t(F^\mu)$ exists for every $t > 0$ and, in addition, has an extension as a function of t to an analytic operator-valued function on $\mathbb{C}_+$ and a strongly continuous function on $\overline{\mathbb{C}_+}$, we say that $J^t(F^\mu)$ exists for all $t \in \overline{\mathbb{C}_+}$. When t is purely imaginary, $J^t(F^\mu)$ is called the analytic operator-valued Feynman integral of F.

The purpose of this paper is to find conditions of μ for which the analytic operator-valued Feynman integral $J^{it}(F^\mu)$ exists.

2. Condition for the existence of analytic operator-valued Feynman integral.

In this section, we establish the existence theorem of the generalized analytic operator - valued Feynman integral under the some conditions.

THEOREM 1. *If $\mathcal{E}_\mu$ is bounded below, densely defined and closed, then there exist a unique, densely defined self-adjoint operator H^μ which is bounded below and satisfies $(H^\mu u, v) = \mathcal{E}_\mu(u,v)$ for all $u \in D(H^\mu)$ and $v \in D(\mathcal{E}_\mu)$*

PROOF. See[1] Theorem 2.6

For $\alpha \geq 0$, μ and ν in $S - S$,$f \in \mathcal{B}(X)$, set of all Borel functions on X, we set

$$U_\nu^{\alpha+\mu} f(x) = E_x[\int_0^\infty e^{-\alpha t - A_t^\mu} f(X_t) dA_t^\nu]$$

provided the right hand side makes sense. When $\nu = m$, we simply write $U^{\alpha+\mu} f$ for $U_\nu^{\alpha+\mu} f$.

Let us introduce the notation, for $\mu \in S - S$, $P_t^\mu f(x) = E_x[e^{-A_t^\mu} f(X_t)]$ provided the right hand side makes sense.

THEOREM 2. *Let $\mu \in S - S$. Then the following assertions are equivalent to each other.*

(1) *$(\mathcal{E}_\mu, \mathcal{F}^\mu)$ is lower semibounded*
(2) *$(P_t^\mu)_{t \geq 0}$ is a strongly continuous semigroup on $L^2(X, m)$*
(3) *There exists constants c and β such that*

$$\|P_t^\mu f\| \leq c e^{\beta t} \|f\|_{L^2(m)}, \forall f \in L^2(m)$$

(4) *There exists $\alpha > 0$ such that*

$$U^{\alpha+\mu}(L^2(m)) \subset L^2(m)$$

(5) *Q_{μ^-} is relatively form bounded with respect to $(\mathcal{E}_{\mu^+}, \mathcal{F}^{\mu^+})$ with bound ≤ 1.*

Furthermore, if any one of the above assertions holds, then the closed quadratic form $(\mathcal{E}_\mu, \mathcal{F}^\mu)$ *corresponding to* $(P_t^\mu)_{t\geq 0}$ *is the largest closed quardratic form that is smaller than* $(\mathcal{E}_\mu, \mathcal{F}^\mu)$.

PROOF. See [1] Theorem 4.1.

From now on, X is the Euclidean space $\mathbb{R}^d$ and m is a Lebesgue measure on $\mathbb{R}^d$. If ψ is a function in $L^2(\mathbb{R}^d, \mathbb{C})$, we denote by ψ_1 its real part and by ψ_2 its imaginary part; i.e., $\psi = \psi_1 + i\psi_2$

THEOREM 3. *Let* $\mu \in S - S$ *be such that*

$$U^{\alpha+\mu}(L^2(m)) \subset L^2(m)$$

for some $\alpha > 0$. *Suppose that* $\mathcal{E}_\mu$ *is closed. If we define* $\mathcal{E}_\mu^C$ *by*

$$\mathcal{E}_\mu^C(\psi, \varphi) = \mathcal{E}_\mu(\psi_1, \varphi_1) + \mathcal{E}_\mu(\psi_2, \varphi_2) + i[\mathcal{E}_\mu(\psi_2, \varphi_1) - \mathcal{E}_\mu(\psi_1, \varphi_2)]$$

for all $\psi, \varphi \in D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu) \subset L^2(\mathbb{R}^d, \mathbb{C})$ *,then* $\mathcal{E}_\mu^C$ *is densely defined, bounded below and closed.*

PROOF. Since $U^{\alpha+\mu}(L^2(m)) \subset L^2(m)$, from Theorem2, $\mathcal{E}_\mu$ is bounded below by A. It suffices to show that $D(\mathcal{E}_\mu^C)$ is complete under the norm

$$|||\psi||| = \sqrt{\mathcal{E}_\mu^C(\psi, \psi) + (-A+1)\|\psi\|^2}.$$

Let (ψ_n) be a sequence in $D(\mathcal{E}_\mu^C)$ such that $|||\psi_n - \psi_m||| \to 0$ as $n, m \to \infty$. For each n, $\psi_n = \psi_{n,1} + i\psi_{n,2}$ where $\psi_{n,1}, \psi_{n,2}$ are in

$D(\mathcal{E}_\mu)$. By the symmetry of $\mathcal{E}_\mu$,

$$\begin{aligned}
&|||\psi_n - \psi_m|||^2 \\
&= \mathcal{E}_\mu^C(\psi_n - \psi_m, \psi_n - \psi_m) + (-A+1)\|\psi_n - \psi_m\|^2 \\
&= \mathcal{E}_\mu(\psi_{n,1} - \psi_{m,1}, \psi_{n,1} - \psi_{m,1}) + \mathcal{E}_\mu(\psi_{n,2} - \psi_{m,2}, \psi_{n,2} - \psi_{m,2}) \\
&\qquad + (-A+1)\|\psi_{n,1} - \psi_{m,1}\|^2 + (-A+1)\|\psi_{n,2} - \psi_{m,2}\|^2 \\
&= \mathcal{E}_\mu(\psi_{n,1} - \psi_{m,1}, \psi_{n,1} - \psi_{m,1}) + (-A+1)\|\psi_{n,1} - \psi_{m,1}\|^2 \\
&\qquad + \mathcal{E}_\mu(\psi_{n,2} - \psi_{m,2}, \psi_{n,2} - \psi_{m,2}) + (-A+1)\|\psi_{n,2} - \psi_{m,2}\|^2 \\
&= |||\psi_{n,1} - \psi_{m,1}|||^2 + |||\psi_{n,2} - \psi_{m,2}|||^2.
\end{aligned}$$

Since $|||\psi_n - \psi_m||| \to 0$, $|||\psi_{n,1} - \psi_{m,1}||| \to 0$ and $|||\psi_{n,2} - \psi_{m,2}||| \to 0$. Since $\mathcal{E}_\mu$ is closed, there exist $\psi_1, \psi_2 \in D(\mathcal{E}_\mu)$ such that $|||\psi_{n,1} - \psi_1||| \to 0$ and $|||\psi_{n,2} - \psi_2||| \to 0$ as $n \to \infty$. Since $\psi = \psi_1 + i\psi_2 \in D(\mathcal{E}_\mu^C)$ and $|||\psi_n - \psi||| \to 0$ as $n \to \infty$, $\mathcal{E}_\mu^C$ is closed.

Let H^μ be a self-adjoint operator given in Theorem 1. If we define H_C^μ on $D(H^\mu) + iD(H^\mu)$ by $H_C^\mu(\psi_1 + i\psi_2) = H^\mu\psi_1 + iH^\mu\psi_2$, then H_C^μ is a self-adjoint operator on $D(H^\mu) + iD(H^\mu) \subset D(\mathcal{E}_\mu^C)$.

THEOREM 4. *Under the conditions of Theorem 3,*

$$(H_C^\mu\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi)$$

for all $\psi \in D(H_C^\mu)$ *and* $\varphi \in (\mathcal{E}_\mu^C)$.

PROOF. By Theorem 1, there exist a unique densely defined self-adjoint operator H^* which is bounded below and satisfies $(H^*\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi)$ for all $\psi \in D(H^*)$ and $\varphi \in (\mathcal{E}_\mu^C)$. From the linearity of H_C^μ, $(H_C^\mu\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi)$ for $\psi = \psi_1 + i\psi_2 \in D(H_C^\mu)$ and $\varphi = \varphi_1 + i\varphi_2 \in D(\mathcal{E}_\mu^C)$. Using consequences [4, $p322$] of the first representation Theorem and the simple fact [5, $p255$] that two self-adjoint operators, one of which extends the other, are actually equal, $H^* = H_C^\mu$.

THEOREM 5. *Let μ be as in Theorem 3. Then the analytic operator valued Feynman integral $J^{it}(F^\mu)$ exists and*

$$J^{it}(F^\mu) = e^{-itH_c^\mu}$$

for all $t \in \mathbb{R}$. Here H^μ is the self - adjoint operator corresponding to $(\mathcal{E}_\mu, \mathcal{F}^\mu)$.

PROOF. Since $(\mathcal{E}_\mu, \mathcal{F}^\mu)$ is a lower semibounded closed quadratic form, we have a self - adjoint operator H^μ which satisfies $(H^\mu u, v) = \mathcal{E}_\mu(u,v)$. Then the semigroup P_t^μ is equal to the semigroup e^{-tH^μ} with generator H^μ so that for every $\phi \in L^2(\mathbb{R}^d)$ and a.e. $x \in \mathbb{R}^d$,

$$e^{-tH^\mu}\phi(x) = E_x[e^{-A_t^\mu(\omega)}\phi(\omega(t))].$$

From the Theorem 3, Theorem 4 and the linearity of E_x,

$$(e^{-tH_c^\mu}\psi)(x) = E_x[e^{-A_t^\mu}\psi(\omega(t))]$$

for every $\psi \in L^2(\mathbb{R}^d, \mathbb{C})$.

It suffices to show that the operator-valued function $A(t) = e^{-tH_C^\mu}$ is defined and strongly continuous for all $t \in \overline{\mathbb{C}_+}$ and is analytic in $\mathbb{C}_+$. For notational simplicity, we let $\mathcal{H} = H_C^\mu$. Since $\mathcal{H}$ is bounded below , $\sigma(\mathcal{H}) \subset [M, +\infty)$ for some $M > -\infty$. Here $\sigma(\mathcal{H})$ denotes the spectrum of $\mathcal{H}$. Since the function $g_t : \sigma(\mathcal{H}) \to \mathbb{C}$ defined by $g_t(u) = e^{-tu}$ is a bounded function of u for any $t \in \overline{\mathbb{C}_+}$, $g_t(\mathcal{H}) = A(t) = e^{-t\mathcal{H}}$ is defined and is a bounded linear operator on $L^2(\mathbb{R}^d, \mathbb{C})$ for all $t \in \overline{\mathbb{C}_+}$ (see.e,g,[5,pp259-264]). Let $\{t_n\}$ be a sequence in $\overline{\mathbb{C}_+}$ such that $t_n \to t$. Then $g_{t_n}(u) \to g_t(u)$ for all $u \in \sigma(\mathcal{H})$. Further,

$$\|g_{t_n}\|_\infty = sup\{|g_{t_n}(u)| : u \in \sigma(\mathcal{H})\}$$

is bounded. Also $g_{t_n}(\mathcal{H}) \to g_t(\mathcal{H})$ strongly, i.e., $A(t_n) = e^{-t_n\mathcal{H}} \to e^{-t\mathcal{H}} = A(t)$ in the strong operator topology as $n \to \infty$.

To show that $A(t) = e^{-t\mathcal{H}}$ is analytic in $\mathbb{C}_+$, it suffices to show that for every $\psi, \varphi \in L^2(\mathbb{R}^d, \mathbb{C})$, $(e^{-t\mathcal{H}}\psi, \varphi)$ is analytic in $\mathbb{C}_+$. We fix $\varphi \in L^2(\mathbb{R}^d, \mathbb{C})$ and we may as well even assume that $\|\varphi\| = 1$. Let P be the spectral measure associated with $\mathcal{H}$. The measure $\mu_{\varphi\varphi}$ defined for any Borel subset B of $\mathbb{R}$ by $\mu_{\varphi\varphi}(B) = (\varphi, P(B)\varphi)$ is a probability measure such that $\mu_{\varphi\varphi}(\sigma(\mathcal{H})) = 1$. Further, by the spectral theorem,

$$(e^{-t\mathcal{H}}\varphi, \varphi) = \int_{\mathbb{R}} e^{-tu}\, d\mu_{\varphi\varphi}(u) = \int_M^\infty e^{-tu}\, d\mu_{\varphi\varphi}(u)$$

Since $e^{-t\mathcal{H}}$ is strongly continuous in $\overline{\mathbb{C}_+}$, it certainly follows that $(e^{-t\mathcal{H}}\varphi, \varphi)$ is continuous in $\mathbb{C}_+$. For any simple closed contour in $\mathbb{C}_+$,

$$\begin{aligned}\int_\Gamma (e^{-t\mathcal{H}}\varphi, \varphi)\, dt &= \int_\Gamma [\int_M^\infty e^{-tu}\, d\mu_{\varphi\varphi}(u)]\, dt \\ &= \int_M^\infty [\int_\Gamma e^{-tu}\, dt]\, d\mu_{\varphi\varphi}(u) \\ &= 0\end{aligned}$$

By the Morera's theorem, $(e^{-t\mathcal{H}}\varphi, \varphi)$ is analytic in $\mathbb{C}_+$. By a polarization argument, $(e^{-t\mathcal{H}}\psi, \varphi)$ is analytic in $\mathbb{C}_+$ for every $\psi, \varphi \in L^2(\mathbb{R}^d, \mathbb{C})$.

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