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SELF-INVOLUTIVE SEMIGROUP

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ABSTRACT. This paper is to study the regular * semigroup, to define the self-involutive semi-group, to introduce the properties of the self-involutive semigroup, and to generalize the maximum idempotent-separating congruence which was found by conditioning self-involutive semigroups.

1. Introduction

A * semigroup is a set S equipped with a binary operation $\cdot$ and unitary operation $* : S \to S$ satisfying the following three axioms :

(1) $(xy)z = x(yz)$,

(2) $(x^*)^* = x$,

(3) $(xy)^* = y^*x^*$ for all $x, y, z \in S$.

Such a unitary operation * is sometimes called an involution, and $(S, \cdot, *)$ is sometimes called an involution semigroup. A semigroup $(S, \cdot)$ is *regular* if for each a in S, there exists x in S such that $a = axa$. When $a = xax$, then a and $y = xax$ are *inverses* in the sense that $aya = a$ and $yay = y$. A *projection* is an impotent e such that $e^* = e$. A * semigroup $(S, \cdot, *)$ is a regular * semigroup if satisfies axiom (4) :

(4) $x = xx^*x$ for all $x \in S$.

Notice that in a regular * semigroup, then x^* is an inverse x since $x^*xx^* = (xx^*x)^* = x^*$ [2].

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EXAMPLE 1.1. Let X be a set and $Y = X \times X$ be the rectangular band on X, i.e., $(x,y)(r,s) = (x,s)$ for every $x,y,r,s \in X$. Define $*$ by $(x,y)^* = (y,x)$. It is easy to check that $(Y,\cdot,*)$ is a regular $*$ semigroup.

Nordahl and Scheiblich [2] proved the following result for regular $*$ semigroups. Since their proof does not use the regularity axiom, it is valid for $*$ semigroups.

THEOREM 1.2. *Let $a \in S$. The map $\sigma : x \to x^*(x \in Ra)$ is a $1:1$ map of Ra onto La^*. The map σ preserves idempotents and H-classes.*

PROOF. If $x \in Ra$, then $xt = a$ and $au = x$ for some $t,u \in S$. Thus, $t^*x^* = a^*$ and $u^*a^* = x^*$ from which $x^* \in La^*$. Similarly, if $(x,y) \in H = R \cap L$, then $(x^*,y^*) \in L \cap R = H$ and so preserves H-classes. If $e \in E$, then $e = ee$ and so $e^* = e^*e^*$, i.e., $e* \in E$. Finally, $\tau : y \to y^*(y \in La^*)$ is an obvious inverse for and so both and are bijective.

THEOREM 1.3. *The involution $*$ fixes one and only one idempotent per R-class.*

PROOF. For $a \in S$, then aa^* is an idempotent element of Ra. Furthermore, $(aa^*)^* = (a^*)^*a^* = aa^*$ and so $*$ fixes at least one idempotent per R-class. Suppose now that $e^2 = e, f^2 = f, (e,f) \in R$, and $e^* = e$. By Theorem 1.2, $(e^*,f^*) \in L$. Since $e = e^*$, then $(e,f^*) \in L$. If $f = f^*$, then $(e,f) \in L \cap R = H$ so $e = f$.

Consider the rectangular band on X, $Y = X \times X$, with $\cdot$ and $*$ defined by $(x,y)(r,s) = (x,s)$ and $(x,y)^* = (y,x)$ for every $x,y,r,s \in X$. Clearly Y is a regular $*$ semigroup and projections are elements of the form (x,x). Note that a multiplication of projections does not always produce a projection : $(x,x)(y,y) = (x,y)$ and $(y,y)(x,x) = (y,x)$. This observation leads to the following two propositions.

PROPOSITION 1.4. *Let S be a * semigroup, and let $e, f \in P$. Then the following statements are equivalent :*

(1) $ef \in P$.
(2) $ef = fe$.
(3) $fef = ef$ *or* $efe = ef$.

PROPOSITION 1.5.

(1) *Let e, f be projections. Then $ef, fe \in E$.*
(2) *For all e, f in P, if $aa^* = a^*a$ for all a in S, then $ef \in P$ (i.e. $ef = fe$).*
(3) *(3) Let S be a regular * semigroup in which every element a commutes its involution a^*. Then S is an orthodox.*
(4) *If S is a^* semigroup and e, f are in P, then efe, fef are in P.*

PROOF. (1) $ef = (ef)(ef)^*(ef) = e(ff)(ee)f = (ef)(ef)$. Thus, $ef \in E$ and dually, $fe \in E$.

(2) Let $e, f \in P$. Then $ef = (ef)(ef)^*(ef) = (ef)(ef)(ef)^* = efeffe = efefe$ and $efe = efefee = efefe$, so $efe = ef$. Thus $(ef)^2 = efef = eff = ef$. And $(ef)^* = (efe)^* = efe = ef$. Hence ef in P.

(3) Let $e, f \in E$. Since $P^2 \subset E$, $ef = (ee^*)(e^*e)(ff^*)(f^*f) = ee^*ff^* \in E$.

(4) $(efe)^2 = efeefe = efe$ and $(efe)^* = efe$. Thus, $efe \in P$. Similarly $fef \in P$.

PROPOSITION 1.6. $E = P^2$.

PROOF. For every $x \in S$, $x^*x, xx^* \in P$. Thus, let $e \in E$, then $e = ee^*e = (ee^*)(e^*e) \in P^2$, and since $P^2 \subset E$ hence $E = P^2$.

PROPOSITION 1.7. *Let S be a regular $*$ semigroup such that $e = xex^* = x^*ex$ for every e in P and for every x in S. Then the projections are central.*

PROOF. For every x in S, xx^* and x^*x are in P. Now $ex = exx^*x = exx^*exx^*x = exx^*ex = exe$ and $xe = xx^*xe = xx^*xex^*xe = xex^*xe = exe$. Hence $ex = xe$.

2. Self-Involutiveness

We now introduce a concept called self-involutiveness and show that the existence of the maximum idempotent-separating congruence for certain type of abelian semigroups.

DEFINITION 2.1. An element x in $*$ semigroup to be *self-involutive* if and only if $x^* = x$. Of course, we define a $*$ semigroup in which all elements are self-involutive will be called a *self-involutive* semigroup

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EXAMPLE 2.2. Let S be the set of all $n \times n$ real diagonal matrices. Let $*$ be the transposition. Clearly $(S, \cdot, *)$ is a self-involutive semigroup.

PROPOSITION 2.3. *Let S be a finite $*$ semigroup with odd number of elements. Then there exists at least one self-involutive element in S*

THEOREM 2.4. *Let S be a $*$ semigroup with 1. Then S has a maximal self-involutive subsemigroup.*

PROOF. Let $F = \{a \in S : a^* = a\}$, and let $\mathcal{C} = \{X \subset F : XX \subset X\}$. Use the set inclusion $\subset$ as a partial ordering on $\mathcal{C}$. Let F be nonempty. Choose $a \in F$. Define $(a) = \{a^n : n \in N\}$. Then clearly $(a) \in \mathcal{C}$ so $\mathcal{C}$ is nonempty. Let $\mathcal{D}$ be a chain in $\mathcal{C}$. Now $\cup\mathcal{D}$ is an upper bound for $\mathcal{D}$ whence there exists a maximal element F_0 in $\mathcal{C}$.

REMARK. We define such F_0 to be a *F-maximal set.* Moreover, F_0 is abelian. Let S be a * semigroup with 1, and let E_0 be a largest $X \subset E$ such that $XX \subset X$. We call such E_0 an *E-maximal set.* Let S be a semigroup, we shall say that an equivalence relation on S is *idempotent-separating* if $\rho \cap (E \times E) = 1_E$, i.e., such that if e, f are idempotents $(e, f) \in \rho$ then $e = f$.

LEMMA 2.5. *Let $\mu[\mu']$ be the maximum idempotent- separating [*] congruence on a regular * semigroup S. Then we have*

$$\mu = \mu' = \{(a, b) \in S \times S : a^*ea = b^*eb \quad \text{and} \quad aea^* = beb^* \quad \text{for all} \quad e \in P\} [2].$$

THEOREM 2.6. *Let S be an abelian semigroup such that $SE \subset E$. Then the maximum idempotent-separating congruence is given by*

$$\rho = \{(a, b) \in S \times S : ae = be, \quad \text{for all} \quad e \in E\}.$$

PROOF. Clearly ρ is an equivalence relation. Let $c \in S$ and $e \in E$. Now $ce \in E$. Let $(a, b) \in \rho$. Then $ace = bce$ and so $(ac, bc) \in \rho$. Hence ρ is right compatible. Similarly ρ is left compatible. Hence ρ is a congruence. If $(e, f) \in \rho \cap (E \times E)$, then $e = ee = fe$ and $f = ff = ef = fe$. So $e = f$. Hence ρ is an idempotent-separating congruence. Finally, let σ be any idempotent-separating congruence, and let $(a, b) \in \sigma$. Since σ is a congruence $(ae, be) \in \sigma$. And since ae and be are idempotents, we have $ae = be$. So $(a, b) \in \rho$. Hence $\sigma \subset \rho$. Thus is the maximum idempotent-separating congruence.

EXAMPLE 2.7. Consider the semigroup $S = \{0, f, a\}$, where

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and where the operation is matrix multiplication. The Cayley table of S is

·	0	f	a
0	0	0	0
f	0	f	0
a	0	0	0

and it is evident that S is an abelian semigroup such that $SE \subset E$. Then the maximum idempotent-separating congruence is

$$\rho = \{(0,0),(f,f),(0,a),(a,0),(a,a)\}.$$

COROLLARY. *Let S be an self-involutive semigroup such that $SE \subset E$. Then the maximum idempotent-separating congruence is given as in Theorem 2.6.*

COROLLARY. *Let μ be the maximum idempotent-separating * congruence on a regular self-involutive semigroup S. Then we have*

$$\mu = \{(a,b) \in S \times S : aea = beb, \quad \text{for all} \quad e \in E\}.$$

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