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THE DENJOY EXTENSION OF THE RIEMANN INTEGRAL

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ABSTRACT. In this paper, we will consider the Denjoy- Riemann integral of functions mapping a closed interval into a Banach space. We will show that a Riemann integrable function on $[a, b]$ is Denjoy-Riemann integrable on $[a, b]$ and that a Denjoy-Riemann integrable function on $[a, b]$ is Denjoy-McShane integrable on $[a, b]$.

1. Introduction

The Denjoy extensions of vector valued integrals have been studied by Gordon [3]. He defined the Denjoy-Dunford, Denjoy -Pettis, and Denjoy-Bochner integrals of functions mapping an interval $[a, b]$ into a Banach space X, which are the extensions of Dunford, Pettis and Bochner integrals, respectively.

He also showed that a Denjoy-Dunford (Denjoy-Bochner) integrable function on $[a, b]$ is Dunford (Bochner) integrable on some subinterval of $[a, b]$ and that for spaces that do not contain a copy of c_0, a Denjoy-Pettis integrable function on $[a, b]$ is Pettis integrable on some subinterval of $[a, b]$.

In this paper, we will study the Denjoy extension of the Riemann integral, called the Denjoy-Riemann integral.

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2. Preliminaries

Throughout this paper, X will denote a Banach space and X^* its dual. Let $F : [a,b] \to X$ be a function and let E be a subset of $[a,b]$. The function F is AC (*absolutely continuous*) on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have points in E and satisfy $\sum_i (d_i - c_i) < \delta$.

The function F is ACG (*generalized absolutely continuous*) on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC.

We recall the following definitions.

DEFINITION 2.1. (a) A function $f : [a,b] \to X$ is *Dunford integrable* on $[a,b]$ if x^*f is Lebesgue integrable on $[a,b]$ for each x^* in X^*. The Dunford integral of f on the measurable set $E \subset [a,b]$ is the vector x_E^{**} in X^{**} such that $x_E^{**}(x^*) = \int_E x^*f$ for all x^* in X^*.

(b) A *McShane partition* of $[a,b]$ is a finite sequence $< ([a_i, b_i], t_i) >_{i \leq n}$ such that $< [a_i, b_i] >_{i \leq n}$ is a non-overlapping family of intervals covering $[a,b]$ and $t_i \in [a,b]$ for each i.

A *gauge* on $[a,b]$ is a function $\delta : [a,b] \to (0, \infty)$. A McShane partition $< ([a_i, b_i], t_i) >_{i \leq n}$ is *subordinate* to a gauge δ if $t_i - \delta(t_i) \leq a_i \leq b_i \leq t_i + \delta(t_i)$ for every $i \leq n$.

A function $f : [a,b] \to X$ is *McShane integrable, with McShane integral* ω, if for every $\epsilon > 0$ there is a gauge $\delta : [a,b] \to (0, \infty)$ such that

$$\|\omega - \sum_{i \leq n} (b_i - a_i) f(t_i)\| < \epsilon$$

for every McShane partition $< ([a_i, b_i], t_i) >_{i \leq n}$ of $[a,b]$ subordinate to δ.

(c) A *tagged partition* of $[a,b]$ is a finite sequence $< ([a_i, b_i], t_i) >_{i \leq n}$ such that $< [a_i, b_i] >_{i \leq n}$ is a non-overlapping family of intervals covering $[a,b]$ and $t_i \in [a_i, b_i]$. A function $f : [a,b] \to X$ is *Riemann integrable* on $[a,b]$ if there exists a vector z in X with the following property : for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|\sum_{i=1}^{n} f(t_i)(b_i - a_i) - z\| < \epsilon$$

whenever $< ([a_i, b_i], t_i) >_{i \leq n}$ is a tagged partition of $[a,b]$ that satisfy $\max\{b_i - a_i : 1 \leq i \leq n\} < \delta$.

Let $F : [a,b] \to X$ and $t \in (a,b)$. The function F is *approximately differentiable* at t if there exists a vector z in X with the following property : there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that

$$\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z$$

for the norm topology of X. In this case, we will write $F'_{ap}(t) = z$.

DEFINITION 2.2. (a) A function $f : [a,b] \to X$ is *Denjoy integrable* on $[a,b]$ if there exists an ACG function $F : [a,b] \to X$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$. The function f is *Denjoy integrable on the set* $E \subset [a,b]$ if $f\chi_E$ is Denjoy integrable on $[a,b]$.

(b) A function $f : [a,b] \to X$ is *Denjoy-McShane integrable* on $[a,b]$ if there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable a.e. on $[a,b]$ and $(x^*F)'_{ap} = x^*f$ a.e. on $[a,b]$.

3. The Denjoy extension of the Riemann integral

In this section, we consider the Denjoy-Riemann integral which is an extension of the Riemann integral.

DEFINITION 3.1. A function $f : [a,b] \to X$ is *Denjoy-Riemann integrable* on $[a,b]$ if there exists an ACG function F such that x^*F is approximately differentiable a.e. and $(x^*F)'_{ap} = x^*f$ a.e. on [a,b].

The following theorem shows that the Denjoy-Riemann integral is an extension of the Riemann integral.

THEOREM 3.2. *If $f : [a,b] \to X$ is Riemann integrable on $[a,b]$, then f is Denjoy-Riemann integrable on $[a,b]$.*

PROOF. Suppose that $f : [a,b] \to X$ is Riemann integrable on $[a,b]$. Let $F(x) = \int_a^x f$. Then by [4, Theorem 8], F is AC on $[a,b]$ and for each $x^* \in X^*$ x^*F is differentiable a.e. on $[a,b]$ and $(x^*F)' = x^*f$ a.e. on $[a,b]$. Hence f is Denjoy-Riemann integrable on $[a,b]$.

The following example shows that there exists a Riemann integrable function that is not measurable.

EXAMPLE 3.3. Define $f : [0,1] \to l_\infty[0,1]$ by $f(t) = \chi_{[0,t]}$. Then since f has no essentially separable range, f cannot be measurable . But f is Riemann integrable on $[0,1]$ by [4, Theorem 9], since f is of outside bounded variation on $[0,1]$.

It follows from Theorem 3.2 and Example 3.3 that a Denjoy-Riemann integrable function is not measurable in general.

THEOREM 3.4. *Let $f : [a,b] \to X$ be a Denjoy-Riemann integrable function and let $F(x) = \int_a^x f$. If F is approximately differentiable a.e. on [a,b], then f is measurable.*

PROOF. Since x^*f is Denjoy integrable for each $x^* \in X^*$, f is weakly measurable [2, Theorem 12]. Since F is continuous on $[a,b]$, F has a separable range. Let Y be the closed linear span of $\{F(t) : t \in [a,b]\}$. Then Y is separable and Y contains the set $\{f(t) : F'_{ap}(t) = f(t)\}$. Hence f is separably valued and f is measurable.

THEOREM 3.5. *If f is Denjoy-Riemann integrable on $[a,b]$, then f is Denjoy- McShane integrable on $[a,b]$.*

PROOF. Suppose that f is Denjoy-Riemann integrable on $[a,b]$. Let $F(x) = \int_a^x f$. Then since F is ACG, x^*F is ACG for each $x^* \in X^*$. By the definition of Denjoy-Riemann integrability, x^*F is approximately differentiable a.e. on $[a,b]$ and $(x^*F)'_{ap} = x^*f$ a.e. on $[a,b]$. Hence f is Denjoy-McShane integrable on $[a,b]$.

THEOREM 3.6. *Let $f : [a,b] \to X$ be Denjoy-Riemann integrable on $[a,b]$ and let $F(x) = \int_a^x f$. Then each perfect set in $[a,b]$ contains a portion on which f is Dunford integrable and F is AC.*

PROOF. Let E be a perfect set in $[a,b]$. Since the function F is ACG on $[a,b]$, by [2,Theorem 4] there exists an interval $[c,d]$ in $[a,b]$ such that $c,d \in [a,b]$, $[c,d] \cap E \neq \emptyset$ and F is AC on $[c,d] \cap E$.

Let $G : [c,d] \to X$ be the function that equals F on $[c,d] \cap E$ and is linear on the intervals contiguous to $[c,d] \cap E$. Then G is AC on $[c,d]$ by [2, Theorem 3]. Since x^*G is AC for each $x^* \in X^*$ on $[c,d]$, x^*G is differentiable a.e. on $[c,d]$ and $(x^*G)' = (x^*F)'_{ap} = x^*f$ a.e. on $[c,d] \cap E$. Hence x^*f is Lebesgue integrable on $[c,d] \cap E$ and f is Dunford integrable on $[c,d] \cap E$.

A function f is a *scalar derivative* of $F : [a,b] \to X$ on $[a,b]$ if for each x^* in X^* the function x^*F is differentiable a.e. on $[a,b]$ and $(x^*F)' = x^*f$ a.e. on $[a,b]$.

COROLLARY 3.7. *Let $f : [a,b] \to X$ be Denjoy-Riemann integrable on $[a,b]$ and let $F(x) = \int_a^x f$. Then there exists a subinterval $[c,d]$ of $[a,b]$ such that f is a scalar derivative of F on $[c,d]$.*

PROOF. Since $[a,b]$ is a perfect set, there exists a subinterval $[c,d]$ of $[a,b]$ on which F is AC by Theorem 3.6.

For each $x^* \in X^*$, x^*F is AC on $[c,d]$ and $(x^*F)'_{ap} = x^*f$ a.e.on [a,b]. By [2,Theorem 27], x^*F is differentiable a.e. on $[c,d]$ and $(x^*F)' = x^*f$ a.e. on $[c,d]$ for each $x^* \in X^*$.

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