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CERTAIN SEQUENCES OF EVALUATION SUBGROUPS

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ABSTRACT. We introduce and study certain subgroups of homotopy groups which contain evaluation subgroups and use these subgroups to obtain a sequence similar to homotopy sequence of fibration. We use the sequence to calculate evaluation subgroups or generalized evaluation subgroups of some topological pair.

1. Introduction

D.H.Gottlieb [1, 2] studied the subgroups $G_n(X)$ of homotopy groups $\pi_n(X)$. In [5, 7, 10], the first author and Woo introduced subgroups $G_n(X,A)$ and $G_n^{Rel}(X,A)$ of $\pi_n(X)$ and $\pi_n(X,A)$ respectively and showed that they fit together into a sequence

$$\begin{aligned}\cdots \to G_n(A) \xrightarrow{i_*} G_n(X,A) \xrightarrow{j_*} G_n^{Rel}(X,A) \xrightarrow{\partial} \cdots \\ \to G_1^{Rel}(X,A) \to G_0(A) \to G_0(X,A)\end{aligned}$$

where i_*, j_* and ∂ are restrictions of the usual homomorphisms of the homotopy sequence

$$\cdots \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \to \cdots \to \pi_0(A) \to \pi_0(X).$$

This sequence are called the G-sequence of (X,A) in [7,10]. We also showed it is exact when the inclusion $i : A \to X$ has a left homotopy inverse or is homotopic to a constant.

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In this paper, we introduce certain subgroups of homotopy groups which contains evaluation subgroups and study its properties. We use these subgroups to obtain a sequence similar to homotopy sequence of a fibration. This sequence will be an important method in the calculations of evaluation subgroups of topological spaces. Actually, we use this sequence to calculate evaluation subgroups of topological spaces.

2. Definitions and Notations

Throughout this paper, all spaces are topological spaces in convinient category which satiesfies exponential law, that is, we always take spaces for which function space exponential law holds. For instance, when we take spaces to be of homotopy type of CW-complexes, this will be holds. Let $[A^A]$ be the subspace of the function space X^A which consists of $f \in X^A$ such that $f(A) \subset A$. Since $[A^A]$ is homeomorphic to A^A, we use A^A instead of $[A^A]$. Let us take s_0 as the base point of S^n and x_0 as the base point of X and its subspaces. We use the same notation ω for the evaluation maps of X^X and X^A into X at the base point x_0 and use i as the inclusion map. And we use X^X and X^A as the path-components of the function spaces X^X and X^A containing 1_X and i respectively. We leave the base points out of the notation for the homotopy groups when the simplication will not lead to confusion. Here we recall the definitions of the evaluation subgroups $G_n(X), G_n^f(X,A)$ of the homotopy group $\pi_n(X)$ and the relative evaluation subgroups $G_n^{Rel}(X,A)$ of the relative homotopy groups $\pi_n(X,A)$ for a CW-pair (X,A) in [2] , [5] and [7,10]. There are two equivalent definitions for each of these subgroups.

DEFINITION 1. $G_n(X) = \omega_*(\pi_n(X^X, 1_X)) = \{[f] \in \pi_n(X) | \exists$ map $H : X \times I^n \longrightarrow X$ such that $[H|_{x_0 \times I^n}] = [f]$ and $H|_{X \times u} = 1_X$ for $u \in \partial I^n\}$[2].

DEFINITION 2. $G_n^h(X,A) = \omega_*(\pi_n(X^A,h)) = \{[f] \in \pi_n(X) |\ \exists$ map $H : A \times I^n \longrightarrow X$ such that $[H|_{x_0 \times I^n}] = [f]$ and $H|_{A\times u} = h$ for $u \in \partial I^n\}$[5].where $h : A \to X$ is a continuous map. If $h : A \to X$ is an inclusion,$G_n^h(X,A)$ is denoted by $G_n(X,A)$

DEFINITION 3. $G_n^{Rel}(X,A) = \omega_*\pi_n(X^A, A^A, i) = \{[f] \in \pi_n(X,A)|$ $\exists$ map $H : (X \times I^n, A \times \partial I^n) \longrightarrow (X,A)$ such that $[H|_{x_0\times I^n}] = [f]$ and $H|_{X\times u} = 1_X$ for $u \in J^{n-1}\}$[7, 10]

These groups fit together into a sequence

$$\xrightarrow{\partial^{n+1}} G_n(A) \xrightarrow{i_*^n} G_n(X,A) \xrightarrow{j_*^n} G_n^{Rel}(X,A) \xrightarrow{\partial^n} \cdots$$
$$\to G_1^{Rel}(X,A) \xrightarrow{\partial^1} G_0(A) \to G_0(X,A)$$

which are called the *G-sequence* of (X,A).

3. Certain subgroups of homotopy groups

We first define certain subgroups of homotopy groups and study those properties. Let $h : X \to Y$ be a continous map and x_0 and y_0 be a base points of X and Y respectively. Let F be the inverse image of y_0 under f, i.e., $F = h^{-1}(y_0)$.

DEFINITION 4. $H = \{[f] \in \pi_n(Y, y_0)|\exists$ map $H : (X \times I^n, F \times \partial I^n) \longrightarrow (Y, y_0)$ such that $[H|_{x_0\times I^n}] = [f]$ and $H|_{X\times u} = h$ for $u \in J^{n-1}\}$ for $n \geq 0$.

Such map H is called the *affiliated map of f with respect to h.* Actually, these subsets are subgroups of homotopy groups $\pi_n(Y, y_0)$. In fact, we can prove that fact by the same way as the proof of that of another evaluation subgroups in section 2. Let Y^X be the function space from X to Y and $y_0(Y^F)$ be the subspace which consists of elements which send F to y_0,i.e., $y_0(Y^F) = \{g \in Y^X | g(F) = y_0\}$ and $\omega : Y^X \to Y$ be the evaluation map at x_0.

THEOREM 1. *$G_n^{hx}(Y,y_0)$ is the image of $\omega_*(\pi_n(Y^X,y_0(Y^F),h))$ under the induced homomorphism of ω, i.e. ,$G_n^{hx}(Y,y_0) = \omega_*(\pi_n(Y^X,y_0(Y^F),h))$.*

PROOF. Suppose $[f] \in G_n^{hx}(Y,y_0)$. Then there is a map $H : (X \times I^n, F \times \partial I^n) \longrightarrow (Y,y_0)$ such that $H|_{x_0 \times I^n} = f$ and $H|_{X \times J^{n-1}} = h$. Define $\widehat{H} : I^n \to Y^X$ by $\widehat{H}(u)(x)$. Then $\widehat{H}(\partial I^n) = H(F \times \partial I^n) = y_0$ and $\widehat{H}(J^{n-1}) = H(x \times J^{n-1}) = h(x)$. Thus the map $\widehat{H}$ is a continuous map from $(I^n, \partial I^n, J^{n-1})$ to $(Y^X, y_0(Y^F), h)$. But

$$\omega\widehat{H}(u) = \widehat{H}(u)(x_0) = \widehat{H}(x_0 \times u) = f(u) \quad \text{for} \quad u \in I^n.$$

Therefore

$$[f] = [\omega\widehat{H}] = \omega_*[\widehat{H}] \in \omega_*(\pi_n(Y^X, y_0(Y^F), h)).$$

Conversely, suppose $[f] \in \omega_*(\pi_n(Y^X,y_0(Y^F),h))$. Then there is an element

$$[g] \in \pi_n(Y^X, y_0(Y^F), h)) \quad \text{such that} \quad \omega_*[g] = [f],$$

where g is a map from $(I^n, \partial I^n, J^{n-1})$ to $(Y^X, y_0(Y^F), h)$. Define $\widetilde{g} : X \times I^n \to Y$ by $\widetilde{g}(x,u) = g(u)(x)$ for $u \in I^n, x \in X$.Then $\widetilde{g}(F \times \partial I^n) = g(\partial I^n)(F) = y_0$ and $\widetilde{g}(x_0 \times I^n) = g(I^n)(x_0) = \omega g(I^n) = y_0$ and $\widetilde{g}(x \times \partial J^{n-1}) = g(J^{n-1})(x) = h(x)$. Thus $\widetilde{g}$ is an affiliated map f w.r.t. h and $[f] \in G_n^{hx}(Y,y_0)$.

In aspect of the Theorem 1, $G_n^{hx}(Y,y_0)$ is called the *evaluation subgroup with respect to h.*

For a continuous map $h : X \to Y$, there are two subgroups of $\pi_n(Y,y_0)$. One is $G_n^{hx}(Y,y_0)$, the other is $G_n^h(Y,X)$ in definition 2. What is the relation between them ? Following theorem shows the relation.

THEOREM 2. *$G_n^{hx}(Y, y_0)$ contains $G_n^h(Y, X)$.*

PROOF. Consider the following commutative diagram;

$$\begin{array}{ccc} (Y^X, h, h,) & \xrightarrow{j} & (Y^X, y_0(Y^F), h) \\ & \searrow_{\omega} \quad {}_{\omega}\swarrow & \\ & (Y, y_0, y_0) & \end{array}$$

where $j : (Y^X, h, h,) \to (Y^X, y_0(Y^F), h)$ is the inclusion. This diagram induces a following commutative diagram;

$$\begin{array}{ccc} \pi_n(Y^X, h, h,) & \xrightarrow{j_*} & \pi_n(Y^X, y_0(Y^F), h) \\ & \omega_* \searrow \qquad \swarrow \omega_* & \\ & \pi_n(Y, y_0, y_0). & \end{array}$$

Thus

$$\begin{aligned} G_n^h(Y, X) &= \omega_*(\pi_n(Y^X, h, h,)) = j_*\omega_*(\pi_n(Y^X, h, h,)) \\ &\subset \omega_*\pi_n(Y^X, y_0(Y^F), h) = G_n^{hx}(Y, y_0). \end{aligned}$$

Corollary 3 follows from Theorem 2.

COROLLARY 3. *$G_n^{hx}(Y, X)$ contains the image of evaluation subgroup of X under h_* induced from $h : X \to Y$.*

PROOF. We know $h_*(G_n(X)) \subset G_n^h(Y, X)$ [5]. By Theorem 2,

$$h_*(G_n(X)) \subset G_n^h(Y, X) \subset G_n^{hx}(Y, X).$$

Let $h : X \to Y$ be a map and $F = h^{-1}(y_0)$. Then $G_n^{hx}(Y, y_0)$ contains the image of the relative evaluation subgroup $G_n^{Rel}(X, F)$ under h_*.

THEOREM 4. $h_*(G_n^{Rel}(X,F)) \subset G_n^{h_X}(Y,y_0)$.

PROOF. Let $[f] \in G_n^{Rel}(X,F)$. Then there is a map

$$H : (X \times I^n, F \times \partial I^n, x_0 \times J^{n-1}) \to (X, F, x_0)$$

such that $H|_{X \times J^{n-1}} = 1_X$ and $H|_{x_0 \times I^n} = f$. Define $\widehat{H}$ by $\widehat{H} = h \circ H$, i.e.,

$$\widehat{H} : (X \times I^n, F \times \partial I^n, x_0 \times J^{n-1}) \xrightarrow{H} (X, F, x_0) \xrightarrow{h} (Y, y_0, y_0)$$

Then $\widehat{H}$ is an affiliated map f w.r.t. h. In fact, $\widehat{H}|_{X \times J^{n-1}} = h \circ H|_{X \times J^{n-1}} = h \circ 1 = h$ and $\widehat{H}|_{x_0 \times I^n} = h \circ H|_{x_0 \times I^n} = h \circ f$. Thus

$$h_*[f] = [h \circ f] \in G_n^{h_X}(Y, y_0).$$

COROLLARY 5. *The evaluation subgroups w.r.t. the identity* $1_X : X \to X$ *are just homotopy groups* $\pi_*(X, x_0)$.

PROOF. Since $F = 1^{-1}(x_0) = x_0$ and $G_n^{Rel}(X,F) = G_n^{Rel}(X,x_0) = \pi_n(X, x_0)$,

$$\pi_n(X,x_0) = G_n^{Rel}(X,F) = 1_* G_n^{Rel}(X,F) \subset G_n^{1_X}(X,x_0) \subset \pi_n(X,x_0).$$

Thus $\pi_n(X,x_0) = G_n^{1_X}(X,x_0)$.

The following theorem shows the relation of two evaluation subgroups w.r.t. two maps respectively which are homotopic to each other.

THEOREM 6. *Let $f, g : X \to Y$ be continuous maps such that $h^{-1}(x_0) = g^{-1}(x_0) = F$. If f is homotopic to g relative to F, then $G_n^{fx}(Y, y_0) = G_n^{gx}(Y, y_0)$.*

PROOF. Let $L : X \times I \to Y$ be a homotopy from f to g relative to F. Consider the adjoint $l : I \to Y^X$ given by $l(t)(x) = L(x,t)$. Then l is a path from f to g. But since $l(t)(u) = L(u,t) = y_0$ for $u \in F$, $l(t)$ belongs to $y_0(Y^F)$ for all $t \in I$. Thus l induces an isomorphism

$$l_\sharp : \pi_n(Y^X, y_0(Y^F), f) \to \pi_n(Y^X, y_0(Y^F), g).$$

Since the diagram

$$\begin{array}{ccc} \pi_n(Y^X, y_0(Y^F), f) & \xrightarrow{l_\sharp} & \pi_n(Y^X, y_0(Y^F), g) \\ \omega_* \big\downarrow & & \big\downarrow \omega_* \\ \pi_n(Y, y_0) & \xrightarrow{1} & \pi_n(Y, y_0) \end{array}$$

is commutative, we have $G_n^{fx}(Y, y_0) = \omega_* \pi_n(Y^X, y_0(Y^F), f) = \omega_* l_\sharp \pi_n(Y^X, y_0(Y^F), f) = \omega_* \pi_n(Y^X, y_0(Y^F), g) = G_n^{gx}(Y, y_0)$.

4. G-sequences of fiberations

In this section, we use evaluation subgroups w.r.t. fibrations to make a sequence similar to homotopy sequence of fibration. We mean fibrations by Hurewicz fibrations

THEOREM 7. *If $p : E \to B$ is a fibration with fiber F, then $p_*(G_n^{Rel}(E, F, e_0)) = G_n^{p_E}(B, b_0)$, where e_0 is the base point in F.*

PROOF. Since $p_*(G_n^{Rel}(E, F)) \subset G_n^{p_E}(B, b_0)$ by Theorem 5, it is sufficient to show that $p_*(G_n^{Rel}(E, F)) \supset G_n^{p_E}(B, b_0)$.

Let $[f] \in G_n^{p_E}(B, b_0)$. Then there is a map $H : (E \times I^n, F \times \partial I^n) \to (B, b_0)$ such that $H|_{E \times J^{n-1}} = p$ and $H|_{e_0 \times I^n} = f$. Define

$H' : E \times I^n \to B$ by $H' = H \circ (1 \times h)$ where $h : (I^n, \partial I^n, I^{n-1} \times 0) \xrightarrow{\cong} (I^n, \partial I^n, J^{n-1})$ is a homeomorphism. Then we have, $H'|_{E \times I^{n-1} \times 0} = H \circ (1 \times h)|_{E \times I^{n-1} \times 0} = H|_{E \times J^{n-1}} = p$, $H'|_{e_0 \times I^n} = H \circ (1 \times h)|_{e_0 \times I^n} = f \circ h$. So we have the following commutative diagram;

$$\begin{array}{ccc} E \times I^{n-1} \times 0 & \xrightarrow{\pi_1} & E \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ E \times I^{n-1} \times I & \xrightarrow{H'} & B \end{array}$$

where π_1 is projection to first axes and i is the inclusion. Since p is a fibration, there is a map $\theta : E \times I^{n-1} \times I \to E$ such that $\theta|_{E \times I^{n-1} \times 0} = \pi_1$ and $p \circ \theta = H'$.Since

$$p \circ \theta(F \times \partial I^n) = H'(F \times \partial I^n) = H \circ (1 \times h)(F \times \partial I^n) = H(F \times \partial I^n) = b_0,$$

we have $\theta(F \times \partial I^n) \subset p^{-1}(b_0) = F$. Moreover,we have $p \circ \theta|_{e_0 \times I^n} = H'|_{e_0 \times I^n} = fh$. Define

$$\bar{\theta} = \theta(1 \times h^{-1}) : E \times I^n \xrightarrow{1 \times h^{-1}} E \times I^n \xrightarrow{\theta} E.$$

Then we have

$$\bar{\theta}(F \times \partial I^n) = \theta(F \times h^{-1}(\partial I^n)) \subset \theta(F \times \partial I^n),$$
$$\bar{\theta}(e, v) = \theta(e, h^{-1}(v)) = \pi_1(e, h^{-1}(v)) = e \quad \text{for} \quad (e, v) \in E \times J^{n-1}.$$

So, $[\bar{\theta}|_{e_0 \times I^n}] \in G_n^{Rel}(E, F)$. But

$$p_*[\bar{\theta}|_{e_0 \times I^n}] = [p\bar{\theta}|_{e_0 \times I^n}] = [p\theta(1 \times h^{-1})|_{e_0 \times I^n}] = [fhh^{-1}] = [f].$$

Therfore, $[f] \in p_*(G_n^{Rel}(E, F))$.

REMARK. Note that if $p : E \to B$ is a serre fibering, then the induced homomorphism $p_* : \pi_n(E, F, e_0) \xrightarrow{\cong} \pi_n(B)$ is one-to-one fashion. Since every fibration (Hurewicz fibering) is Serre fibering, there is a following long exact sequence called the homotopy sequence of fibering for given fibration $p : E \to B$ with fiber F;

$$\cdots \to \pi_n(F) \to \pi_n(E) \xrightarrow{j_*} \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

We would like to obtain a subsequence of the homotopy sequence of fibering similar to G-sequence. Consider the following diagram which consists of the G-sequence of the pair (E,F) and the evaluation subgroup of $\pi_n(B, b_0)$ w.r.t. $p : E \to B$;

$$\begin{array}{ccccccccc} \to G_n(F) & \longrightarrow & G_n(E,F) & \xrightarrow{j_*} & G_n^{Rel}(E,F) & \xrightarrow{\partial} & G_{n-1}(F) & \longrightarrow \\ & & & & \Big\downarrow{\scriptstyle p_*} & & & \\ & & & & G_n^{p_E}(B, b_0) & & & \end{array}$$

Define $\bar{j}_* = p_* \circ j_*$ and $\bar{\partial} = \partial \circ p_*^{-1}$. Then $\bar{j}_*$ and $\bar{\partial}$ are well-defined because p_* is bijective by Theorem 7 and remark. Consequently, we obtain following theorem.

THEOREM 8. *If $p : E \to B$ is a fibration with fiber F, then we have the following sequence;*

$$\to G_n(F) \longrightarrow G_n(E,F) \xrightarrow{\bar{j}_*} G_n^{p_E}(B, b_0) \xrightarrow{\bar{\partial}} G_{n-1}(F) \longrightarrow$$

The sequence in Theorem 8 is called the *G-sequence of fibration p*. Corollary 9 follows from the definition immediately.

COROLLARY 9. *The G-sequence of fibration $p : E \to B$ with fiber F is exact if the pair (E,F) has the exact G-sequence.*

REMARK. Note that every locally trivial bundle with paracompact base space is a fibration(Hurewicz fibering). The identification η_n : $S^{2n+1} \to CP^n$ is fibration because it is locally trivial bundle with fiber S^1 and CP^n is paracompact.

LEMMA 10. *If A is a subcomplex of S^n, then the pair (S^n, A) has an exact G-sequence.*

PROOF. The inclusion $i : A \to S^n$ has a left homotopy inverse or is null homotopic. By Theorem 3.7, 3.10 in [10], (S^n, A) has an exact G-sequence.

THEOREM 11. *The evaluation subgroup $G_k^{\eta_n s^{2n+1}}(CP^n)$ with repect to η_n is isomorphic to $\pi_k(CP^n)$ for $k > 2$.*

PROOF. By Lemma 10 and above remark, we have the following exact G-sequence of the fibration $\eta_n : S^{2n+1} \to CP^n$ with fiber S^1;

$$\to G_k(S^1) \to G_k(S^{2n+1}, S^1) \to G_k^{\eta_n s^{2n+1}}(CP^n) \to G_{k-1}(S^1) \to .$$

Since $G_k(S^{2n+1}, S^1) = \pi_k(S^{2n+1})$ and $G_l(S^1) = 0$ for $k > 1$, we have

$$G_k^{\eta_n s^{2n+1}}(CP^n) \cong G_k(S^{2n+1}, S^1) = \pi_k(S^{2n+1}) = \pi_k(CP^n)$$

for $k > 2$.

REFERENCES

1. D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840–856.

2. ________, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–756.
3. ________, *Covering transformations and universal fibrations*, Illinois J. Math. **13** (1969), 432-437.
4. B. Gray, *Homotopy Theory*, Academic press, New York, 1975.
5. J. R. Kim and M. H. Woo, *Certain subgroups of homotopy groups*, J. of Korean Math. Soc. **21** (1984), 109–120.
6. G. E. Lang, Jr., *Evaluation subgroups of factor spaces*, Pacific J. Math. **42** (1972), 701–709.
7. K. Y. Lee and M. H. Woo, *The relative evaluation subgroups of a CW-pair*, J. of Korean Math. Soc. **25** (1988), 149–160.
8. ________, *Homology and generalized evaluation subgroups of homotopy groups*, J. of Korean Math. Soc. **25** (1988), 333–342.
9. ________, *Exact G-sequences and relative G-pairs*, J. of Korean Math. Soc. **27** (1990), 177–184.
10. ________, *G-sequences and ω-homology of a CW-pair*, Topology and its application **52** (1993), 221–236.
11. J. Siegel, *G-spaces W-spaces H-spaces*, Pacific J. Math. **31** (1969), 209–214.
12. G. W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, New York, 1978.

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