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V–SEMICYCLIC MAPS AND FUNCTION SPACES

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ABSTRACT. For any map $v : X \to Y$, the generalized Gottlieb set $G(\Sigma A; X, v, Y)$ with respect to v is a subgroup of $[\Sigma A, Y]$. If $v : X \to Y$ has a left homotopy inverse $u : Y \to X$, then for any $f \in G(\Sigma A; X, v, Y), g \in G(\Sigma A; Y, u, X)$, the function spaces $L(\Sigma A, X; uf)$ and $L(\Sigma A, X; g)$ have the same homotopy type.

1. Introduction

This work is a continuation of the study of the Gottlieb set $G(A, Y)$ developed by Gottlieb[2,3] and Varadarajan[7]. Oda[6] generalized $G(A, Y)$ to $G(A, X, v, Y)$. In this paper we study some properties of $G(A; X, v, Y)$ and define a generalization of $G(A; X, v, Y)$ when A is a co-H-group. In section 2, we show that for any map $v : X \to Y$, $G(A; X, v, Y)$ is a subgroup of $[A, Y]$ when A is a co-H-group. We define a v-semicyclic pair which is a generalized concept of a v-cyclic map and obtain some sufficient conditions for a pair of maps is to be a v-semicyclic pair. In section 3, we show that $G(A; X, v, Y)$ is the image of the induced map of evaluation map from fuction space $L(X, Y; v)$ to Y, and obtain a sufficient condition for homotopy equivalence of components of $L(\Sigma A, X)$.

Throughout this paper, space means a space of homotopy type of locally finite connected CW complex. We assume also that spaces

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have non-degenerate base points. All maps shall mean continuous functions. The base point as well as the constant map will be denoted by $*$. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The folding map $\nabla : X \vee X \to X$ is given by$\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. ΣX denotes the reduced suspension of X. Frequently $j : X \vee Y \to X \times Y$ will be reserved for the inclusion.

2. v-cyclic maps and v-semicyclic pairs

In this section we show that for any map $v : X \to Y$, $G(A; X, v, Y)$ is a subgroup of $[A, Y]$ when A is a co-H-group. We define a v-semicyclic pair which is a generalized concept of a v-cyclic map and obtain some sufficient conditions for a pair of maps is to be a v-semicyclic pair.

DEFINITION 2.1[7]. A map $f : A \to X$ is called *cyclic* if there exists a map $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. Since $j : X \vee A \to X \times A$ is a cofibration, this is equivalent to saying that we can find a map $G : X \times A \to X$ such that $Gj = \nabla(1 \vee f)$. The set of all homotopy classes of cyclic maps from A to X is denoted by $G(A, X)$.

DEFINITION 2.2[6]. Let $v : X \to Y$ be a map. A map $f : A \to Y$ is called *v-cyclic* if there is a map $F : X \times A \to Y$ such that $Fj \sim \nabla(v \vee f) : X \vee A \to Y$, where $j : X \vee A \to X \times A$ is the inclusion. It is clear that $f : A \to Y$ is v-cyclic if and only if $v : X \to Y$ is f-cyclic. *The generalized Gottlieb set $G(A; X, v, Y)$ with respect to $v : A \to X$* is the set of all homotopy classes of v-cyclic maps from A to Y. It is clear that if $u \sim v : X \to Y$, then $G(A; X, u, Y) = G(A; X, v, Y)$. If

$v \sim 1_X : X \to X$, then $G(A; X, v, X)$ is just the Gottlieb set $G(A, X)$.

REMARK 2.3. In general, $G(A,Y) \subsetneqq G(A; X, v, Y)$. It is known[3] that $G(S^2, S^2) = 0$. Let $\eta : S^3 \to S^2$ be the Hopf map. Then there is a map $F : S^2 \times S^3 \to S^2$ such that $Fj \sim \nabla(\iota \vee \eta)$. Thus we know, from Proposition 2.4, that $G(S^2; S^3, \eta, S^2) = \pi_2(S^2) = \mathbb{Z}$.

PROPOSITION 2.4. *If $v : X \to Y$ is cyclic, then $G(A; X, v, Y) = [A, Y]$.*

PROOF. Let $f : A \to Y$ be any map. Since $v : X \to Y$ is cyclic, there is a map $V : Y \times X \to Y$ such that $V(\ ,*) \sim 1$ and $V(*,\ ) \sim v$. Consider the map $F = V(f \times 1)T : X \times A \xrightarrow{T} A \times X \xrightarrow{(f\times 1)} Y \times X \xrightarrow{V} Y$, where $T : X \times A \to A \times X$ is given by $(x, a) = (a, x)$. Then $F(\ ,*) \sim v$ and $F(*,\ ) \sim f$. Thus $f \in G(A; X, v, Y)$.

THEOREM 2.5. *Let $v : X \to Y$ be a map. Then $G(A; X, v, Y) = \bigcap_u G(A; B, vu, Y)$ for any space B and any map $u : B \to X$. In particular, $G(A, Y) = G(A; Y, 1, Y) = \bigcap_u G(A; B, u, Y)$ for any space B and any map $u : B \to Y$.*

PROOF. Let $f \in G(A; X, v, Y)$. Then there is a map $F : X \times A \to Y$ such that $Fj \sim \nabla(v \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion. For any space B and any map $u : B \to X$, let $G : B \times A \to Y$ be the composition $B \times A \xrightarrow{u\times 1} X \times A \xrightarrow{F} Y$. Then $Gj' = F(u \times 1)j' = Fj(u \vee 1) \sim \nabla(vu \vee f)$, where $j' : B \vee A \to B \times A$ is the inclusion. Thus $f \in G(A; B, vu, Y)$ for any space B and any map $u : B \to X$ and $f \in \bigcap_u G(A; B, vu, Y)$ for any space B and any map $u : B \to X$. On the other hand, let $f \in \bigcap_u G(A; B, vu, Y)$ for any space B and any map $u : B \to X$. Take $B = X$ and $u = 1_X : X \to X$. Then $f \in G(A; X, v, Y)$. This proves the theorem.

COROLLARY 2.6. *If $v : X \to Y$ has a right homotopy inverse $u : Y \to X$, then $G(A; X, v, Y) = G(A, Y)$.*

LEMMA 2.7. *Let $v : X \to Y$ be a map. If $f : A \to Y$ is a v-cyclic map and $\theta : B \to A$ is an arbitrary map, then $f\theta : B \to Y$ is a v-cyclic map.*

PROOF. Let $F : X \times A \to Y$ be a map such that $F(*, \ ) \sim f$ and $F(\ , *) \sim v$. Then $G = F(1 \times \theta) : X \times B \to Y$ satisfies $G(*, \ ) \sim f\theta$ and $G(\ , *) \sim v$. This proves the lemma.

Let f and g be pointed maps from A to Y where A is a co-H-space with μ be a co-H-structure. Define $f + g : A \to Y$ to be the composition

$$A \xrightarrow{\mu} A \vee A \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y.$$

THEOREM 2.8. *Let A be a co-H-group. For any map $v : X \to Y$, $G(A; X, v, Y)$ is a subgroup of $[A, Y]$.*

PROOF. Let $\mu : A \to A \vee A$ be a co-H-structure and $\nu : A \to A$ be the inversion on A. Let $f, g \in G(A; X, v, Y)$. Then there are maps $F : X \times A \to Y, G : X \times A \to Y$ such that $F(x, *) = v(x), F(*, a) = f(a)$ and $G(x, *) = v(x), G(*, a) = g(a)$. Let $J : X \times (A \vee A) \to Y$ be given by $J(x, a, *) = F(x, a), J(x, *, a) = G(x, a)$. Since $F(x, *) = v(x) = G(x, *)$, J is well defined and continuous. Also it is clear that $J(*, \mu(a)) = (\nabla(f \vee g)\mu)(a)$. Define a map $H : X \times A \to Y$ by the composition

$$X \times A \xrightarrow{1 \times \mu} X \times (A \vee A) \xrightarrow{J} Y.$$

Then $H(x, *) = J(x, *, *) = v(x)$ and $H(*, a) = J(*, \mu(a)) = (\nabla(f \vee g)\mu)(a)$. Thus we know that $f + g \in G(A; X, v, Y)$. From Lemma 2.7, we know that $f\nu : A \to Y$ is v-cyclic. This proves the theorem.

DEFINITION 2.9. Let $v : X \to Y$ be a map and A a co-H-space. A pointed map $f : A \to Y$ is called *v-semicyclic* if there exists a map $f' : A \to Y$ such that $f + f'$ is v-cyclic. Then a pair $\{f, f'\}$ is said to be a *v-semicyclic pair* in $[A, Y]$. The set of all homotopy classes of v-semicyclic maps from A to Y is denoted by $SG(A; X, v, Y)$. In particular, $SG(S^n; X, v, Y)$ will be denoted by $SG_n(X, v, Y)$.

REMARK 2.10. Let $v : X \to Y$ a map and A a co-H-space. Let $f : A \to Y$ be a v-cyclic map. Since $f \sim f + * : A \to Y$, $f : A \to Y$ is a v-semicyclic map. Thus we know that $G(A; X, v, Y) \subset SG(A; X, v, Y)$. In general, $G(A; X, v, Y) \subsetneqq SG(A; X, v, Y)$. It is known[10] that $1_{S^5} : S^5 \to S^5$ is not a cyclic map, but $1_{S^5} + 1_{S^5} : S^5 \to S^5$ is a cyclic map. Thus we know that $1_{S^5} \in SG_5(S^5, 1_{S^5}, S^5)$, but $1_{S^5} \notin G_5(S^5, 1_{S^5}, S^5)$.

From the definition of v-semicyclic pair and Theorem 2.8, we have the following proposition.

PROPOSITION 2.11. *Let $v : X \to Y$ be a map and A a co-H-space. If $\{f, f'\}$ and $\{g, g'\}$ are v-semicyclic pairs in $[A, Y]$, then $\{f + f' + g, g'\}$, $\{f, f' + g + g'\}$ and $\{f'^{-1}, f^{-1}\}$ are v-semicyclic pairs in $[A, Y]$.*

THEOREM 2.12. *Let $v : X \to Y$ and $u : Y \to Z$ be maps ,and A a co-H-space. If $f \in G(A; X, v, Y)$ and $g \in G(A; Y, u, Z)$, then $\{uf, g\}$ is a uv-semicyclic pair in $[A, Z]$.*

PROOF. There are maps $F : X \times A \to Y$ and $G : Y \times A \to Z$ such that $Fj \sim \nabla_Y(v \vee f)$ and $Gi \sim \nabla_Z(u \vee g)$ respectively, where $j : X \vee A \to X \times A, i : Y \vee A \to Y \times A$ are inclusions. Let $\mu : A \to A \vee A$ be a co-H-structure on A. Consider the map $H : X \times A \to Z$ to be the composition

$$X \times A \xrightarrow{1\times\mu} X \times (A \vee A) \xrightarrow{1\times j'} X \times A \times A \xrightarrow{F\times 1} Y \times A \xrightarrow{G} Z,$$

where $j' : A \vee A \to A \times A$ is the inclusion. Then $Hj = G(F \times 1)(1 \times j')(1 \times \mu)j \sim \nabla_Z(u \vee g)(\nabla_Y(v \vee f) \vee 1)(1 \vee \mu) = \nabla_Z(1 \vee \nabla_Z)(uv \vee uf \vee g)(1 \vee \mu) = \nabla_Z(uv \vee \nabla_Z(uf \vee g)\mu)$. Thus we know that $uf + g \in G(A; X, uv, Z)$. This proves the theorem.

COROLLARY 2.13. *Let A be a co-H-space.*

(1) *If $f \in G(A; X, v, Y)$ and $g \in G(A, Y) = G(A; Y, 1, Y)$, then $\{f, g\}$ is a v-semicyclic pair in $[A, Y]$.*
(2) *If $f \in G(A, X)$ and $g \in G(A; X, v, Y)$, then $\{vf, g\}$ is a v-semicyclic pair in $[A, Y]$.*
(3) *Let $v : X \to Y$ has a left homotopy inverse $u : Y \to X$. If $f \in G(A; X, v, Y)$ and $g \in G(A; Y, u, X)$, then $\{uf, g\}$ is 1_X-semicyclic pair in $[A, X]$.*

We also, as a corollary, have the following lemma.

LEMMA 2.14. *Let $u : Y \to Z$ be a map. Then $u_*(G(A; X, v, Y)) \subset G(A; X, uv, Z)$.*

THEOREM 2.15. *Let $u : Y \to Z$ has a left homotopy inverse. Then $G(A; X, uv, Z) \cap u_*([A, Y]) = u_*(G(A; X, v, Y))$.*

PROOF. It follows, from Lemma 2.14, that $u_*(G(A; X, v, Y)) \subset G(A; X, uv, Z) \cap u_*([A, Y])$. Conversely, let $f \in G(A; X, uv, Z) \cap u_*([A, Y])$. Then there are maps $g : A \to Y$ and $F : X \times A \to Z$ such that $u_*(g) = ug \sim f$ and $Fj \sim \nabla(uv \vee f)$ respectively. Define a map $G : X \times A \to X$ by $G(x, a) = wF(x, a)$, where $w : Z \to Y$ is a left homotopy inverse of $u : Y \to Z$. Then $G(\ ,*) = wF(\ ,*) \sim wuv \sim v$ and $G(*,\ ) = wF(*,\ ) \sim wf \sim w(ug) = (wu)g \sim g$. Thus $g \in G(A; X, v, Y)$ and $f = u_*(g) \in u_*(G(A; X, v, Y))$. This proves the theorem.

3. Evaluation fibrations of function spaces

In this section we show that $G(A;X,v,Y)$ is the image of the induced map of evaluation map from fuction space $L(X,Y;v)$ to Y, and obtain a sufficient condition for homotopy equivalence of components of $L(\Sigma A,X)$. From now on, let A be a compact CW complex. We use the following notations. $L(A,X)$ will denote the spaces of maps from A to X with the compact open topology and $L(A,X;f)$ the path component of $L(A,X)$ containing $f: A \to X$. $L_0(A,X)$ and $L_0(A,X;f)$ will denote the space of base point preserving maps in $L(A,X)$ and $L(A,X;f)$ respectively. According to a well known fact, $L(A,X)$ and $L_0(A,X)$ have the homotopy type of CW complexes. Clearly the evaluation map $\omega : L(A,X) \to X$ is a fibration. Let $f: A \to X$ be a pointed map. Since X is a path connected, the restriction $\omega_f = p|_{L(A,X;f)} : L(A,X;f) \to X$ is a fibration with fiber $L_0(A,X;f)$. We call this fibration $(L(A,X;f),\omega_f,X)$ *the evaluation fibration defined by f.* First we recall the following well-known lemma.

LEMMA 3.1. *Let X be a locally compact Hausdorff space, Z a Hausdorff space and Y any space. Then the function spaces $L(Z,L(X,Y))$ and $L(X\times Z,Y)$ are homeomorphic and a homeomorphism $H : L(Z,L(X,Y)) \to L(X\times Z,Y)$ is given by $H(g)(x,z) = g(z)(x)$ for each $g: Z\to L(X,Y), x\in X, z\in Z$. Furthermore, $f\sim g$ iff $H(f)\sim H(g)$.*

THEOREM 3.2. *Let $\omega : L(X,Y;v) \to Y$ be the evaluation fibration. Then $\omega_*([A,L(X,Y;v)]) = G(A;X,v,Y)$ as set, where ω_* is the induced function of ω.*

PROOF. Since X is a locally compact, any continuous map $h: (A,*) \to (L(X,Y;v),v)$ gives rise to a continuous map $H(h): X\times A \to Y$. Since $H(h)(*,a) = h(a)(*) = \omega h(a)$ and $H(h)(x,*) = h(*)(a) = v(a)$, we have $\omega_*([h]) = [\omega h] \in G(A;X,v,Y)$. Conversely, let $[f] \in G(A; X, v, Y)$. Then there is a map $F : X \times A \to Y$ such that $F_{|X} = v$ and $F_{|A} = f$. Define $g : A \to L(X, v, Y)$ by $g(a)(x) = H^{-1}(F)(x, a)$. Since $g(*)(x) = H^{-1}(F)(x, *) = v(x)$ and $\omega g(a) = g(a)(*) = H^{-1}(F)(*, a) = f(a)$, $[f] = [\omega g] = \omega_*([g]) \in \omega_*[A, L(X, Y; v)]$. This completes the theorem.

Under the same hypotheses as the above theorem, if, in addition, A is a co-H-group, then $\omega_*([A, L(X, Y; v]) = G(A; X, v, Y)$ as groups.

THEOREM 3.3. *Let $\omega_f : L(A, Y; f) \to Y$ be the evaluation fibration and $v : X \to Y$ a map. Then there exists a map $s_f : X \to L(A, Y; f)$ such that $\omega_f s_f \sim v$ if and only if $f : A \to Y$ is v-cyclic.*

PROOF. Let $s_f : X \to L(A, Y; f)$ be a map such that $\omega_f s_f = v$. Define a map $H : X \times A \to Y$ by letting $H(x, a) = s_f(x)(a)$. Then $H : X \times A \to Y$ is a continuous map and $H(x, *) = s_f(x)(*) = \omega_f s_f(x) = v(x), H(*, a) = s_f(*)(a)$. Since $s_f(*)$ belongs to $L_0(A, Y; f)$, $s_f(*)$ is homotopic to f. Thus $f : A \to Y$ is v-cyclic. On the other hand, suppose that $f : A \to Y$ is v-cyclic. Then there is a map $F : X \times A \to Y$ such that $Fj = \nabla(v \vee f)$. Define a map $s_f : X \to L(A, X; f)$ by letting $s_f(x)(a) = F(x, a)$. Since X is a path connected space, $s_f : X \to L(A, X; f)$ is a map such that $\omega_f s_f = v$. This completes the theorem.

For any $f \in [A, Y]$ the evaluation map $\omega_f : L(A, Y; f) \to Y$ is a fibration with fibre $L_0(A, Y; f)$. Then we have a long exact sequence of homotopy groups

$$\begin{aligned}\cdots \to [\Sigma^{r+1} B, Y] &\xrightarrow{\partial} [\Sigma^r B, L_0(A, Y; f)] \\ &\xrightarrow{i_*} [\Sigma^r B, L(A, Y; f)] \xrightarrow{\omega_*} [\Sigma^r B, Y] \xrightarrow{\partial} \cdots .\end{aligned}$$

It is known[4] that for $f \in L_0(\Sigma A, Y), L_0(\Sigma A, Y; *)$ is homotopy equivalent to $L_0(\Sigma A, Y; f)$ and $[\Sigma(B \wedge A), Y]$ is isomorphic to

$[B, L_0(\Sigma A, Y; *)]$. From the above fact, and Theorem 3.2 and Theorem 3.3, we have the following corollary.

COROLLARY 3.4.

(1) *There is a short exact sequence*

$$0 \to [\Sigma(\Sigma^r B \wedge A), Y] \xrightarrow{i_*} [\Sigma^r B, L(\Sigma A, Y; f)] \xrightarrow{\omega_*} G(\Sigma^r B; \Sigma A, f, Y) \to 0.$$

(2) *If $f : A \to Y$ is a v-cyclic map for a surjection $v : X \to Y$, then $[\Sigma^r B, Y] = G(\Sigma^r B; A, f, Y)$ for $r \geq 1$.*

We showed[10] that the following lemma.

LEMMA 3.5[10]. *If $f + g : \Sigma A \to X$ is cyclic, then the evaluation fibrations $(L(\Sigma A, X; f), \omega_f, X)$ and $(L(\Sigma A, X; g), \omega_g, X)$ are fibre homotopy equivalent.*

Combining Corollary 2.13 and Lemma 3.5, we have the following theorem.

THEOREM 3.6. *Let $v : X \to Y$ has a left homotopy inverse $u : Y \to X$. Then for any $f \in G(\Sigma A; X, v, Y)$ and $g \in G(\Sigma A; Y, u, X)$, the evaluation fibrations $(L(\Sigma A, X; uf), \omega_{uf}, X)$ and $(L(\Sigma A, X; g), \omega_g, X)$ are fibre homotopy equivalent.*

LEMMA 3.7[1]. *$[f, g] = 0$ if and only if there is a map $m : \Sigma A \times \Sigma B \to X$ such that $mj \sim \nabla(f \vee g)$.*

THEOREM 3.8. *Let $f \in G(\Sigma A; X, v, Y)$. Then for any map $g : \Sigma B \to X$, $[v_*(g), f] = 0$ in $[\Sigma(A \wedge B), Y]$.*

PROOF. Since $f \in G(\Sigma A; X, v, Y)$, there is a map $F : X \times \Sigma A \to Y$ such that $Fj \sim \nabla(v \vee f)$. Define a map $m : \Sigma B \times \Sigma A \to Y$ to be the composition

$$\Sigma B \times \Sigma A \xrightarrow{(g \times 1)} X \times \Sigma A \xrightarrow{F} Y.$$

Then $mj' = F(g \times 1)j' = Fj(g \vee 1) \sim \nabla(vg \vee f)$. By Lemma 3.7, we have $[v_*(g), f] = 0$.

We recall (Proposition 3.4 in[1]) one more fact regarding the generalized Whitehead product. If A and B are themselves suspensions and $f, \bar{f} \in [\Sigma A, X]$ and $g, \bar{g} \in [\Sigma B, X]$, then

(1) $[f + \bar{f}, g] = [f, g] + [\bar{f}, g]$.
(2) $[f, g + \bar{g}] = [f, g] + [f, \bar{g}]$.

COROLLARY 3.9. *Let $\{f, f'\}$ be a v-semicyclic pair in $[\Sigma A, Y]$. Then for any map $g : \Sigma B \to X$, $[v_*(g), f] = -[v_*(g), f']$.*

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