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REDUCIBILITY OF DIFFERENTIAL EQUATIONS

SE-MOK SONG

ABSTRACT. We obtain some properties of reducible differential equations in the sense of Liapunov.

1. Preliminaries

Consider the linear differential equation

$$x' = A(t)x, \tag{1}$$

where the coefficient matrix $A(t)$ is continuous on the real line $\mathbb{R}$.

The equation (1) is said to be *kinematically similar* to another equation

$$y' = B(t)y \tag{2}$$

(we write (1) $\overset{k}{\sim}$ (2)) if there exists a bounded continuously differentiable invertible matrix $S(t)$ with bounded inverse $S^{-1}(t)$ such that if $x(t)$ is any solution of (1) then $S(t)x(t)$ is a solution of (2). In this case we can write

$$B(t) = S(t)A(t)S^{-1}(t) + S'(t)S^{-1}(t) \tag{3}$$

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because

$$\begin{aligned}\frac{d}{dt}[S(t)x(t)] &= S'(t)x(t) + S(t)x'(t)\\ &= [S'(t) + S(t)A(t)]x(t)\\ &= B(t)S(t)x(t)\end{aligned}$$

implies that

$$S'(t) + S(t)A(t) = B(t)S(t).$$

We can show that the kinematic similarity is an equivalence relation (Lemma 1).

Coppel's reducibility definition[2] is as follows: The equation (1) is said to be *reducible* if it is kinematically similar to (2) whose coefficient matrix $B(t)$ has the block form

$$\begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix}$$

where $B_1(t)$ and $B_2(t)$ being matrices of lower order than $B(t)$.

Although this definition agrees with the definition of reducibility in linear algebra, it differs from Liapunov's use of the term: The equation (1) is *reducible* if it is kinematically similar to a linear system with constant coefficient.

In this paper we will obtain some properties of reducible differential equations in the sense of Liapunov. To do this we need the following concept.

Consider the nonautonomous differential equations

$$x' = f(t, x), \tag{4}$$

$$y' = g(t, y), \tag{5}$$

where $f, g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are C^1 functions. Suppose that (4) and (5) have a unique solution for any initial value $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, and any solution of (4) and (5) exists on $\mathbb{R}$.

For any real column vector x, $||x||$ will denote the norm. The same symbol will be used for the compatible matrix norm.

We say that (4) and (5) are *topologically equivalent*(we write (4) $\sim$ (5)) if there exists a continuous function $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying

(i) $\lim_{||x|| \to \infty} ||H(t,x)|| = \infty$ uniformly in $t \in \mathbb{R}$ and $\lim_{||x|| \to ||x_0||} ||H(t,x)|| = ||H(t,x_0)||$ uniformly in $t \in \mathbb{R}$ for any $x_0 \in \mathbb{R}^n$,

(ii) for fixed $t \in \mathbb{R}, H_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by $H_t(x) = H(t,x)$ is a homeomorphism,

(iii) $G_t = H_t^{-1}$ has property (i) also,

(iv) if $x(t)$ is any solution of (4) then $H(t, x(t))$ is a solution of (5), and if $y(t)$ is any solution of (5) then $G(t, y(t))$ is a solution of (4).

Also, we can define the topological equivalence between two linear systems

$$x' = A(t)x \tag{1}$$

$$y' = B(t)y. \tag{2}$$

Palmer[3] characterized exponential dichotomy in terms of topological equivalence.

2. Main Results

LEMMA 1. *Kinematic similarity is an equivalence relation.*

PROOF. For the reflexivity, we take $S(t) = I$, the identity matrix. If (1) $\overset{k}{\sim}$ (2), then

$$B(t) = [S(t)A(t) + S'(t)]S^{-1}(t)$$

by the relation (3). Thus we have

$$A(t) = [T(t)B(t) + T'(t)]T^{-1}(t),$$

where $T(t) = S^{-1}(t)$. It follows that (2) $\overset{k}{\sim}$ (1).

Now suppose that (1) $\overset{k}{\sim}$ (2) and (2) $\overset{k}{\sim}$ (6), where

$$z' = C(t)z. \tag{6}$$

Then we have

$$B(t) = [S(t)A(t) + S'(t)]S^{-1}(t)$$

and

$$C(t) = [U(t)A(t) + U'(t)]U^{-1}(t)$$

for some $U(t)$. Therefore if we put $V(t) = S(t)U(t)$, then

$$C(t) = [V(t)A(t) + V'(t)]V^{-1}(t).$$

This implies that (1) $\overset{k}{\sim}$ (6).

LEMMA 2. *Kinematic similarity implies topological equivalence.*

PROOF. Assume that (1) $\overset{k}{\sim}$ (2). Then we have

$$A(t) = S^{-1}(t)B(t)S(t) - S^{-1}(t)S'(t).$$

We can assume that $|S(t)|, |S^{-1}(t)| \leq K$ for some constant $K > 0$. Define $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$H(t, x) = S(t)x.$$

Clearly, H is continuous. We show that (1) $\sim$ (2) via $H(t, x)$.

(i) $\lim_{||x||\to\infty} ||H(t,x)|| = \infty$ uniformly in $t \in \mathbb{R}$: Suppose not. Then There exist a constant $M > 0$, sequences $k_m \to \infty, (t_m)$ in $\mathbb{R}$ and (x_m) in $\mathbb{R}^n$ such that $||x_m|| > k_m$ and $||H(t_m, x_m)|| = ||S(t_m)x_m|| \le M$. Thus we have

$$\begin{aligned} ||x_m|| &= ||S^{-1}(t_m)S(t_m)x_m|| \\ &\le ||S^{-1}(t_m)||\ ||S(t_m)x_m|| \\ &\le KM. \end{aligned}$$

This is a contradiction since $||x_m|| > k_m \to \infty$.

$$\lim_{||x||\to||x_0||} ||H(t,x)|| = ||H(t,x_0)|| \quad \text{uniformly in } \ t \in \mathbb{R}:$$

This follows from the following computation.

$$\begin{aligned} ||H(t,x) - H(t,x_0)|| &= ||S(t)x - S(t)x_0|| \\ &\le ||S(t)||\ ||x - x_0|| \\ &\le K||x - x_0||. \end{aligned}$$

(ii) Note that $H_t(x) = H(t,x) = S(t)x$ and $S(t)$ is invertible. Hence H_t is a homeomorphism for each $t \in \mathbb{R}$.

(iii) $G_t(y) = G(t,y) = S^{-1}(t)y = H_t^{-1}(y)$ has property (i) also.

(iv) We show that $H(t,x(t))$ is a solution of (2) when $x(t)$ is any solution of (1).

$$\begin{aligned} \frac{d}{dt}H(t,x(t)) &= \frac{d}{dt}[S(t)x(t)] = S'(t)x(t) + S(t)x'(t) \\ &= S'(t)x(t) + S(t)A(t)x(t) \\ &= [S'(t)S^{-1}(t) + S(t)A(t)S^{-1}(t)]S(t)x(t) \\ &= B(t)S(t)x(t), B(t) = S'(t)S^{-1}(t) + S(t)A(t)S^{-1}(t) \\ &= B(t)H(t,x(t)). \end{aligned}$$

Hence $H(t,x(t))$ is a solution of (2). Similarly, we can show that $G(t,y(t))$ is a solution of (1) if $y(t)$ is any solution of (2). This completes the proof.

THEOREM 3. *Boundedness is an invariance property under the topological equivalence map H.*

PROOF. Assume that (4) $\sim$(5) via $H(t,x)$. Let $x(t)$ be a bounded solution of (4), say $||x(t)|| \leq M$ for some constant $M > 0$. We prove that $||H(t,x(t))|| < K$ for some constant $K > 0$.

Assume the contrary. Then there are sequences (t_m) in $\mathbb{R}$ and (x_m) in $\mathbb{R}^n$ such that

$$||x_m|| \leq M \quad \text{and} \quad ||H(t_m,x_m)|| \to \infty \quad \text{as } m \to \infty.$$

Therefore we have

$$||G(t_m, H(t_m,x_m))|| \to \infty \quad \text{as } m \to \infty.$$

which is a contradiction since

$$||G(t_m, H(t_m,x_m))|| = ||x_m|| \leq M.$$

THEOREM 4. *If there exists a constant $T > 0$ such that*

$$A(t+T) = A(t) \quad \textit{for all } t \in \mathbb{R},$$

then the equation (1), that is, the T-periodic equation, is topologically equivalent to a linear system with constant coefficient.

PROOF. By the Floquet theorem [1, Theorem 1.1.24], there exist a differentiable T-periodic matrix function $P(t)$ and a constant matrix R such that the equation (1) becomes

$$y' = Ry \tag{7}$$

by the transformation $x = P(t)y$. Then we can prove that (1) $\sim$ (7) via $H(t,x) = P(t)x$, by the similar manner in the proof of Lemma 2.

THEOREM 5. *If the equation (1) is reducible, then there are a homeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ and a linear homeomorphism C satisfying*

$$T = H_t^{-1} \circ C \circ H_0. \tag{8}$$

PROOF. Let $X(t, t_0, x_0)$ be a solution of (1) satisfying $X(t_0, t_0, x_0) = x_0$. By the assumption, the equation (1) is topologically equivalent to

$$y' = By \tag{9}$$

where B is the constant matrix, via $H(t, x)$. Since $H(t, X(t, t_0, x_0))$ is a solution of (9), we have

$$H(t, X(t, t_0, x_0)) = e^{Bt} y_0$$

for some $y_0 \in \mathbb{R}^n$. If we put $t = t_0$, then we obtain

$$H(t_0, x_0) = H(t, X(t_0, t_0, x_0)) = e^{Bt} y_0,$$

that is, $y_0 = e^{-Bt_0} H(t_0, x_0)$. Thus we have

$$H(t, X(t, t_0, x_0)) = e^{B(t-t_0)} H(t_0, x_0).$$

In particular

$$H(t, X(t, 0, x_0)) = e^{Bt} H(0, x_0) = e^{Bt} H_0(x_0)$$

when $t_0 = 0$. Hence

$$\begin{aligned} H_t[T(x_0)] &= H_t(X(t, 0, x_0)) \\ &= e^{Bt}[H_0(x_0)] = C[H_0(x_0)], \end{aligned}$$

where T is a homeomorphism $x_0 \mapsto X(t, 0, x_0)$ and $C = e^{Bt}$. Consequently we have

$$H_t \circ T = C \circ H_0.$$

THEOREM 6. *If the equation (1) is reducible, then we obtain a formula*

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} f(t,x) = BT(t,x) \tag{10}$$

for some homeomorphism $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ and a constant matrix B.

PROOF. We have (1) $\sim$ (9) via $H(t,x)$. Define $T(t,x) = H(t,x)$. Then, since $T(t, X(t,t_0,x_0))$ is a solution of (9),

$$\begin{aligned}\frac{\partial}{\partial t} T(t, X(t,t_0,x_0)) + \frac{\partial}{\partial x} T(t, X(t,t_0,x_0)) f(t, X(t,t_0,x_0)) \\ = BT(t, X(t,t_0,x_0)).\end{aligned}$$

This implies that

$$\frac{\partial}{\partial t} T(t_0,x_0) + \frac{\partial}{\partial x} T(t_0,x_0) f(t_0,x_0) = BT(t_0,x_0)$$

when $t = t_0$. Therefore we have the relation (10).

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DEPARTMENT OF MATHEMATICS EDUCATION
CHONGJU UNIVERSITY
CHONGJU 360-764, KOREA