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ESSENTIAL SEQUENCES AND GENERALIZED FRACTIONS

SANG-CHO CHUNG AND DONG-SOO LEE†

ABSTRACT. We investigate associated prime ideals of the module of generalized fractions defined by poor essential sequences and extend the McAdam and Ratliff's criterion of locally unmixed rings.

1. Introduction

In 1983, Ratliff [R] introduced the asymptotic sequence in Noetherian rings and characterized locally quasi-unmixed rings. Furthermore, in 1985, he and McAdam [MR] gave an analogous concept of the asymptotic sequence, that is, they defined the essential sequence (see below Definition) which is an asymptotic sequence itself under Noetherian rings and obtained a criterion of locally unmixed rings.

For a module M of a commutative ring R, in 1982, Sharp and Zakeri defined a triangular subset $U_n(\subset R^n)$ and, using this set, constructed the module of generalized fractions $U_n^{-n}M$ which is a generalization of localizations of modules.

On the other hand, when R is Noetherian, we [CL] got the following associated prime ideals of modules of generalized fractions $(U_a)_n^{-n}R$ induced by the triangular subset $(U_a)_n$ consisting of poor asymptotic

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sequences and obtained an extended criterion of locally quasi-unmixed rings. We had the following

$$\begin{aligned}&\{\mathfrak{p} \in \mathrm{Spec}(R) : a\text{-}grade(\mathfrak{p}) = ht\ \mathfrak{p} = n-1\}\\ \subset\ &\mathrm{Ass}((U_a)_n^{-n}R)\\ \subset\ &\{\mathfrak{p} \in \mathrm{Spec}(R) : a\text{-}grade(\mathfrak{p}) = a\text{-}grade(\mathfrak{p}R_{\mathfrak{p}}) = n-1\}.\end{aligned}$$

and the next equivalent conditions

(1) R is locally quasi-unmixed.
(2) $a\text{-}grade(I) = ht\ I$ for all ideals I in R.
(3) $a\text{-}grade(\mathfrak{m}) = ht\ \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ in R.
(4) If I is an ideal of the principal class in R, then $a\text{-}grade(I) = ht\ I$.
(5) $\mathrm{Ass}((U_a)_n^{-n}R) = \{\mathfrak{p} \in \mathrm{Spec}(R) : ht\ \mathfrak{p} = n-1\}$ for all $n = 1, 2, \ldots$.
(6) The following complex defined in [RSZ, p. 52]

$$0 \longrightarrow R \longrightarrow (U_a)_1^{-1}R \longrightarrow \cdots \longrightarrow (U_a)_i^{-i}R \longrightarrow \cdots$$

is of Cousin type for R with respect to the height filtration $\mathcal{F} = (F_i)_{i\geq 0}$ where $F_i = \{\mathfrak{p} \in \mathrm{Spec}(R) : ht\ \mathfrak{p} \geq i\}$ (see [RSZ, 3.1]).

Suppose that R is a Noetherian ring. The purpose of this paper is to study associated prime ideals of the modules of generalized fractions defined by poor essential sequences (see below Definition) and to give an extended criterion of locally unmixed rings by means of associated prime ideals of the above modules. We show that the set $(U_e)_n$ whose members are poor essential sequences is a triangular subset (Lemma 1), and hence this triangular subset provides a module of generalized fractions $(U_e)_n^{-n}R$ by [SZ1]. Next we obtain the bounds of associated prime ideals of $(U_e)_n^{-n}R$ (Theorem 3) and an extended criterion of locally unmixed rings (Corollary 4), *i.e.*, we have the following dual conclusions, these are

$$\begin{aligned} &\{\mathfrak{p} \in \operatorname{Spec}(R) : e\text{-}grade(\mathfrak{p}) = ht\ \mathfrak{p} = n-1\} \\ \subset &\operatorname{Ass}((U_e)_n^{-n}R) \\ \subset &\{\mathfrak{p} \in \operatorname{Spec}(R) : e\text{-}grade(\mathfrak{p}) = e\text{-}grade(\mathfrak{p}R_{\mathfrak{p}}) = n-1\}. \end{aligned}$$

and the next equivalent conditions

(1) R is locally unmixed.
(2) $e\text{-}grade(I) = ht\ I$ for all ideals I in R.
(3) $e\text{-}grade(\mathfrak{m}) = ht\ \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ in R.
(4) If I is an ideal of the principal class in R, then $e\text{-}grade(I) = ht\ I$.
(5) If $a_1, \ldots, a_n$ are an essential sequence in R and $\mathfrak{p} \in E((a_1, \ldots, a_n)R)$, then $ht\ \mathfrak{p} = n$.
(6) $\operatorname{Ass}((U_e)_n^{-n}R) = \{\mathfrak{p} \in \operatorname{Spec}(R) : ht\ \mathfrak{p} = n-1\}$ for all $n = 1, 2, \ldots$.

2. Definitions and Results

Throughout this note, R is a commutative Noetherian ring with non-zero identity, and if R is local then we denote R^* the completion of R. Let $(a_1, \ldots, a_n)R$ be the ideal of R which is generated by the set of elements $\{a_1, \ldots, a_n\}$ in R.

We call a ring R *locally unmixed* if, for each maximal ideal $\mathfrak{m}$, $dim((R_{\mathfrak{m}})^*/\mathfrak{z}) = dim(R_{\mathfrak{m}})^*$ for all $\mathfrak{z} \in \operatorname{Ass}(R_{\mathfrak{m}})^*$, and *unmixed* if R is local.

DEFINITION. [MR] Let $\mathfrak{a}$ and $\mathfrak{p}$ be ideals in R such that $\mathfrak{p}$ is prime. Then $\mathfrak{p}$ is an *essential prime divisor* of $\mathfrak{a}$ in case $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{p}(R_{\mathfrak{p}})^*$ is minimal over $\mathfrak{a}(R_\mathfrak{p})^* + \mathfrak{z}$ for some $\mathfrak{z} \in \mathrm{Ass}(R_\mathfrak{p})^*$. The set of essential prime divisors of $\mathfrak{a}$ will be denoted $E(\mathfrak{a})$.

The elements $a_1, \ldots, a_n$ of R is said to be a *poor essential sequence* if, for $i = 1, \ldots, n$, $a_i \notin \bigcup_{\mathfrak{p} \in E((a_1,\ldots,a_{i-1})R)} \mathfrak{p}$; it is said to be an *essential sequence* if, in addition, $(a_1, \ldots, a_n)R \neq R$.

If $\mathfrak{a}$ is an ideal of R, then the *essential grade* of $\mathfrak{a}$, denoted *e-grade*($\mathfrak{a}$), is the length of a maximal essential sequences in $\mathfrak{a}$ and we interpret *e-grade*$(R) = \infty$.

LEMMA 1. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of R. Then we have*

$$e\text{-}grade(\mathfrak{a}\mathfrak{b}) = e\text{-}grade(\mathfrak{a} \cap \mathfrak{b}) = min\{e\text{-}grade(\mathfrak{a}), e\text{-}grade(\mathfrak{b})\}.$$

PROOF. By [MR, 4.3], for an ideal I of R, we have

$$e\text{-}grade(I) = min\{dim((R_\mathfrak{p})^*/\mathfrak{z}) : \mathfrak{z} \in \mathrm{Ass}(R_\mathfrak{p})^* \text{ with } I \subset \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

On the other hand, since R is Noetherian, every ideal has a primary decomposition. Therefore this completes the proof.

PROPOSITION 2. *Let $(U_e)_n = \{(a_1, \ldots, a_n) \in R^n : a_1, \ldots, a_n$ is a poor essential sequence in R such that if $a_i = 1$ for some $i = 1, \ldots, n-1$ then $a_j = 1$ for all $j (\geq i)\}$. Then $(U_e)_n$ is a triangular subset of R^n.*

PROOF. We show that $(U_e)_n$ satisfies the three conditions of triangular subset [SZ1].

Clearly $(U_e)_n \neq \emptyset$, since $(1, \ldots, 1) \in (U_e)_n$.

Next, let $(a_1, \ldots, a_n) \in (U_e)_n$. Then we may assume that *e-grade* $(a_1, \ldots, a_n)R = n$. Hence by [MR, 5.5] we have $(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}) \in (U_e)_n$ for some positive integers α_i.

Finally, let $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in (U_e)_n$. Also we may assume that $e\text{-}grade(a_1, \ldots, a_n)R = n$ and $e\text{-}grade(b_1, \ldots, b_n)R = n$. Then by Lemma 1, there is $(c_1, \ldots, c_n) \in (U_e)_n$ such that for each $i = 1, \ldots, n$,

$$(c_1, \ldots, c_i)R \subset (a_1, \ldots, a_i)R \cap (b_1, \ldots, b_i)R.$$

THEOREM 3. *Fix a non-negative integer* n *such that* $n \leq \sup_{\mathfrak{m} \in \mathrm{Max}(R)} e\text{-}grade(\mathfrak{m})$. *Put*

$(U_e)_{n+1} = \{(a_1, \ldots, a_{n+1}) \in R^{n+1} : a_1, \ldots, a_{n+1}$ *is a poor essential sequence in* R *such that for some* $i(1 \leq i < n)$ *if* $a_i = 1$ *then* $a_j = 1$ *for all* $j \geq i\}$ *and*

$(U_e)_n = \{(a_1, \ldots, a_n) \in R^n :$ *there is* $a_{n+1} \in R$ *such that* $(a_1, \ldots, a_{n+1}) \in (U_e)_{n+1}\}$. *Consider the following conditions.*

$A_n = \{\mathfrak{p} \in \mathrm{Spec}(R) : e\text{-}grade(\mathfrak{p}) = ht\ \mathfrak{p} = n\}$.

$A'_n = \{\mathfrak{p} \in \mathrm{Spec}(R) : R_\mathfrak{p}$ *is unmixed such that* $e\text{-}grade(\mathfrak{p}) = e\text{-}grade(\mathfrak{p}R_\mathfrak{p}) = n\}$.

$B_n = \{\mathfrak{p} \in \mathrm{Spec}(R) : e\text{-}grade(\mathfrak{p}) = e\text{-}grade(\mathfrak{p}R_\mathfrak{p}) = n\}$.

$B'_n = \{\mathfrak{p} \in \mathrm{Spec}(R) :$ *for some* $(a_1, \ldots, a_n) \in (U_e)_n$ $\mathfrak{p} \in E((a_1, \ldots, a_n)R)\}$.

Then we have $A_n = A'_n$, $B_n = B'_n$ *and*

$$A_n \subset \mathrm{Ass}((U_e)_{n+1}^{-n-1}R) \subset B_n.$$

PROOF. The first and the second assertions follow immediately from [MR, 6.1, 5.7 and 5.10].

For the third assertion, we show that the first inclusion holds. Let $\mathfrak{p} \in A_n$ hence $ht\ \mathfrak{p} = n$. Let $\Phi : R \longrightarrow R_\mathfrak{p}$ be the natural map. Then

$$\Phi(U_e)_{n+1} = \{(\Phi(a_1), \ldots, \Phi(a_{n+1})) \in R_\mathfrak{p}^{n+1} : (a_1, \ldots, a_{n+1}) \in (U_e)_{n+1}\}$$

is a triangular subset of $R_{\mathfrak{p}}^{n+1}$ and $((U_e)_{n+1}^{-n-1}R)_{\mathfrak{p}} = \Phi(U_e)_{n+1}^{-n-1}R_{\mathfrak{p}}$ by [HS, 2.1]. Since $\mathrm{Supp}((U_e)_{n+1}^{-n-1}R) \subset \{\mathfrak{q} \in \mathrm{Spec}(R) : ht\, \mathfrak{q} \geq n\}$ by [HS, 3.1], it is enough to show that $\Phi(U_e)_{n+1}^{-n-1}R_{\mathfrak{p}} \neq 0$. Since $e\text{-}grade(\mathfrak{p}) = e\text{-}grade(\mathfrak{p}R_{\mathfrak{p}}) = dim\ R_{\mathfrak{p}} = n$ and $R_{\mathfrak{p}}$ is unmixed, we may assume that

$$\begin{aligned}\Phi(U_e)_{n+1} = \{(\Phi(a_1), \ldots, \Phi(a_{n+1})) \in R_{\mathfrak{p}}^{n+1} : &\text{ there exists } j \text{ with}\\ &0 \leq j \leq n \text{ such that } \Phi(a_1), \ldots, \Phi(a_j) \text{ form a system of}\\ &\text{parameters for } R_{\mathfrak{p}} \text{ and } \Phi(a_{j+1}) = \cdots = \Phi(a_{n+1}) = 1\}\end{aligned}$$

; for, $ht(a_1, \ldots, a_{n+1})R \geq n+1$ and if $\Phi(a_i) = 1$ for some $i < n+1$, then by [SZ1, 3.3] we have $\dfrac{r}{(\Phi(a_1), \ldots, \Phi(a_{n+1}))} = 0$.

Therefore by [SZ2, 3.5] and [BH, 3.5.7] we have

$$\Phi(U_e)_{n+1}^{-n-1}R_{\mathfrak{p}} \cong \mathbf{H}^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0.$$

Next for the second inclusion, let $\mathfrak{p} \in \mathrm{Ass}((U_e)_{n+1}^{-n-1}R)$. Then by [SZ3, 5.1] there is $\dfrac{r}{(a_1, \ldots, a_n, 1)} \in (U_e)_{n+1}^{-n-1}R$ such that

$$\left(0 : \frac{r}{(a_1, \ldots, a_n, 1)}\right) = \mathfrak{p}.$$

Since $(a_1, \ldots, a_n)R \subset \mathfrak{p}$ by [SZ1, 3.3], we have $e\text{-}grade(\mathfrak{p}) \geq n$. Suppose that $e\text{-}grade(\mathfrak{p}) > n$. Then for all $\mathfrak{q} \in E((a_1, \ldots, a_n)R)$ we have $e\text{-}grade(\mathfrak{q}) = n$ by [MR, 5.10]. Hence there is $a_{n+1} \in \mathfrak{p} \setminus \bigcup_{\mathfrak{q} \in E((a_1, \ldots, a_n)R)} \mathfrak{q}$ such that

$$(a_1, \ldots, a_{n+1}) \in (U_e)_{n+1}.$$

Therefore we have

$$\left(0 : \frac{r}{(a_1, \ldots, a_n, 1)}\right) = \left(0 : \frac{r}{(a_1, \ldots, a_{n+1})}\right) = \mathfrak{p}$$

by [SZ3, 5.1] again. Hence we have the following contradiction.

$$\frac{a_{n+1}r}{(a_1,\ldots,a_{n+1})} = \frac{r}{(a_1,\ldots,a_n,1)} = 0.$$

On the other hand, by [M, Corollary(p. 38)] we get

$$\mathfrak{p} \in \mathrm{Ass}((U_e)_{n+1}^{-n-1}R) \Longleftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}(\Phi(U_e)_{n+1}^{-n-1}R_{\mathfrak{p}}).$$

Note that, for all $(a_1,\ldots,a_{n+1}) \in (U_e)_{n+1}$ we have *e-grade*$((\Phi(a_1),\ldots,\Phi(a_i))R) \geq i$ for $i = 1,\ldots,n$ by [MR, 5.7.1]. Hence replacing R by $R_{\mathfrak{p}}$, we have *e-grade*$(\mathfrak{p}R_{\mathfrak{p}}) = n$, using the same argument as above.

COROLLARY 4. *The following statements are equivalent.*

(1) *R is locally unmixed.*
(2) *e-grade$(I) = ht\ I$ for all ideals I in R.*
(3) *e-grade$(\mathfrak{m}) = ht\ \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ in R.*
(4) *If I is an ideal of the principal class in R, then e-grade$(I) = ht\ I$.*
(5) *If $a_1,\ldots,a_n$ are an essential sequence in R and $\mathfrak{p} \in E((a_1,\ldots,a_n)R)$, then $ht\ \mathfrak{p} = n$. (For $n = 0$ we take this to mean that $\mathfrak{p} \in E((0)) = \mathrm{Ass}(R)$ implies $ht\ \mathfrak{p} = 0$.)*
(6) *$\mathrm{Ass}((U_e)_{n+1}^{-n-1}R) = \{\mathfrak{p} \in \mathrm{Spec}(R) : ht\ \mathfrak{p} = n\}$ for all $n = 0,1,2,\ldots$.*

PROOF. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) These are immediate conclusions of [MR, 6.1].

(2) $\Rightarrow$ (6) By the hypothesis we have *e-grade*$(\mathfrak{p}) = ht\ \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Therefore the proof follows from Theorem 3.

(6) $\Rightarrow$ (3) Suppose that $ht\ \mathfrak{m} = n$ for some $\mathfrak{m} \in \mathrm{Max}(R)$. Then $\mathfrak{m} \in \mathrm{Ass}((U_e)_{n+1}^{-n-1}R)$ by the hypothesis. Hence by Theorem 3 we have *e-grade*$(\mathfrak{m}) = n$.

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Department of Mathematics
Chungnam National University
Taejon 305-764, Korea