JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 9, July 1996

TRIANGULAR SUBSETS AND COASSOCIATED PRIME IDEALS

SANG-CHO CHUNG

ABSTRACT. We study the relationship between a property of a triangular subset and coassociated prime ideals of the module of generalized fractions induced by the triangular subset, and investigate coassociated prime ideals of modules of generalized fractions defined by some special triangular subsets.

1. Introduction

Throughout this note, R is a commutative Noetherian ring with non-zero identity and M is an R-module.

In 1995, Yassemi[Y] introduced the coassociated prime ideal (definition 4) which is the dual concept of the associated prime ideal. It is very useful to study the new properties of non-Noetherian and non-Artinian module.

In 1982, Sharp and Zakeri[SZ1] gave the module of generalized fractions $U_n^{-n}M$ (Definition 1) of a given R-module M with respect to a triangular subset U_n which is the generalization of the localization of modules. In general, even though the given R-module is finitely generated, the induced module of generalized fractions is non-Noetherian and non-Artinian.

So we interest the coassociated prime ideals of modules of generalized fractions. We obtain the relationship between a property of

The author was partially supported by BSRI, the Ministry of Education, Project No. 95-1427.

Received by the editors on May 7, 1996.

1991 *Mathematics subject classifications*: Primary 13C05.

a triangular subset and coassociated prime ideals of the module of generalized fractions induced by the triangular subset(Theorem 7), and give coassociated prime ideals(Theorem 9) of modules of generalized fractions defined by some special triangular subsets(Example 2). That is, we have

(1) $a_1 \notin \bigcup_{\mathfrak{p} \in \mathrm{Coass}(U_n^{-n}M)} \mathfrak{p}$ for each $(a_1, \ldots, a_n) \in U_n$.
(2) $\mathrm{Coass}(U_h)_n^{-n}M \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : ht_M\mathfrak{p} = 0\}$.
(3) $\mathrm{Coass}(U_s)_n^{-n}M \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : dim\ M/\mathfrak{p}M = d\}$, where $dim\ M = d$.
(4) $\mathrm{Coass}(U_r)_n^{-n}M \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : depth(\mathfrak{p}, M) = 0\}$.
(5) If, in addition, R is local, then we have $\mathrm{Coass}(U_f)_n^{-n}M \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : depth_{R_\mathfrak{p}} M_\mathfrak{p} = 0\}$.

In [Y, 1.7], Yassemi proved that, for an R-module M, $\mathfrak{p} \in \mathrm{Coass}(M)$ if and only if there exists $\mathfrak{m} \in \mathrm{Max}(R) \cap V(\mathfrak{p})$ such that $\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}(M, E(R/\mathfrak{m})))$. Hence when we investigate coassociated prime ideal, it is valuable to investigate $\mathrm{Ann}(\mathrm{Hom}(M, E(R/\mathfrak{m})))$. For this reason, Theorem 10 (if $\mathrm{Ann}(x) \subset \mathfrak{m}$ for all non-zero $x \in M$, then $\mathrm{Ann}_R M = \mathrm{Ann}\big(\mathrm{Hom}(M, E(R/\mathfrak{m}))\big)$ which is an extended version of Sharp's result is help to find coassociated prime ideals.

2. Preliminaries and Main Results

We use ^{T} to denote matrix transpose, n to denote a positive integer, and $D_n(R)$ to denote the set of $n \times n$ lower triangular matrices over R. For $H \in D_n(R)$, $|H|$ denotes the determinant of H. Let $(a_1, \ldots, a_i)R$ be the ideal of R which is generated by $\{a_1, \ldots, a_i\}$ and let $(a_1, \ldots, a_i)M$ be the submodule of M which is generated by $\{a_j m : j = 1, \ldots, i \text{ and } m \in M\}$.

DEFINITION 1. [SZ1] A *triangular subset of* R^n is a non-empty subset U_n of R^n such that

(i) if $(a_1, \ldots, a_n) \in U_n$, then $(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}) \in U_n$ for all choices of positive integers $\alpha_1, \ldots, \alpha_n$, and

(ii) if $(a_1, \ldots, a_n) \in U_n$ and $(b_1, \ldots, b_n) \in U_n$, then there exist $(c_1, \ldots, c_n) \in U_n$ and $H, K \in D_n(R)$ such that $H[a_1 \ \ldots \ a_n]^T = [c_1 \ \ldots \ c_n]^T = K[b_1 \ \ldots \ b_n]^T$.

Let U_n be a triangular subset of R^n. The *module of generalized fractions $U_n^{-n}M$ of M with respect to U_n* is a module, whose elements, called *generalized fractions*, have the form

$$\frac{m}{(a_1, \ldots, a_n)},$$

where $m \in M$ and $(a_1, \ldots, a_n) \in U_n$, satisfying the following condition.

Let $m, m', s \in M$ and $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in U_n$. Then $\dfrac{m}{(a_1, \ldots, a_n)} = \dfrac{m'}{(b_1, \ldots, b_n)}$ if and only if there exist $(c_1, \ldots, c_n) \in U_n$ and $H, K \in D_n(R)$ such that $H[a_1 \ \ldots \ a_n]^T = [c_1 \ \ldots \ c_n]^T = K[b_1 \ \ldots \ b_n]^T$ and $|H|m - |K|m' \in (c_1, \ldots, c_{n-1})M$.

The addition and scalar multiplication in U_n^{-n} are such that

$$\frac{m}{(a_1, \ldots, a_n)} + \frac{s}{(b_1, \ldots, b_n)} = \frac{|H|m + |K|s}{(c_1, \ldots, c_n)}$$

for any choice of $(c_1, \ldots, c_n) \in U_n$ and $H, K \in D_n(R)$ such that $H[a_1 \ \ldots \ a_n]^T = [c_1 \ \ldots \ c_n]^T = K[b_1 \ \ldots \ b_n]^T$, and

$$r \cdot \frac{m}{(a_1, \ldots, a_n)} = \frac{rm}{(a_1, \ldots, a_n)}$$

for $r \in R$. The reader is referred to [SZ1, SZ2] for more details of the construction.

When $M \neq 0$, *dim* M (or $dim_R M$) denotes the dimension of M, that is the supremum of lengths of chains of prime ideals of $\mathrm{Supp}(M)$.

For $\mathfrak{p} \in \text{Supp}(M)$, the M-height of $\mathfrak{p}$, denoted by $ht_M\mathfrak{p}$, is defined to be $dim_{R_\mathfrak{p}} M_\mathfrak{p}$ and $ht_M(a_1, \ldots, a_n)R$ is defined to be $\inf\{ht_M\mathfrak{p} : (a_1, \ldots, a_n)R \subset \mathfrak{p}\}$. We interpret $ht_M R = \infty$.

EXAMPLE 2. Let R be a Noetherian ring. Then the following non-empty sets are triangular subsets of R^n.

(1) (cf. [SZ2], 5.2) Suppose that M is a finitely generated R-module.
$(U_h)_n = \{(a_1, \ldots, a_n) \in R^n : ht_M(a_1, \ldots, a_i)R \geq i \quad (1 \leq i \leq n)\ \}$.

(2) ([C], 1.1) Suppose that M is a finitely generated R-module of dimension d.
$(U_s)_n = \{(a_1, \ldots, a_n) \in R^n : dim\ M/(a_1, \ldots, a_i)M = d - i \quad (1 \leq i \leq n)\}$.

(3) ([SZ1], 3.10) Let M be a finitely generated R-module.
$(U_r)_n = \{(a_1, \ldots, a_n) \in R^n : a_1, \ldots, a_n$ forms a poor M-sequence $\}$.

(4) ([C], 1.2) Suppose that $(R,\ \mathfrak{m})$ is a local ring and M is a finitely generated R-module.
$(U_f)_n = \{(a_1, \ldots, a_n) \in R^n : \dfrac{a_1}{1}, \ldots, \dfrac{a_n}{1}$ in $R_\mathfrak{p}$ forms an $M_\mathfrak{p}$-sequence for all $\mathfrak{p} \in \text{Supp}(M) \backslash \{\mathfrak{m}\}$ such that $(a_1, \ldots, a_n)R \subset \mathfrak{p}\}$.

REMARK 3. [C] The above triangular subsets give modules of generalized fractions and we introduced their associated prime ideals. That is

(1) $\text{Ass}\,((U_h)_n^{-n}M) = \{\mathfrak{p} \in \text{Supp}(M) : ht_M\mathfrak{p} = n - 1\}$.
(2) $\text{Ass}\,((U_s)_n^{-n}M) = \{\mathfrak{p} \in \text{Supp}(M) : ht_M\mathfrak{p} = n - 1$ and $dim\ M/\mathfrak{p}M = d - n + 1\}$.
(3) $\text{Ass}\,((U_r)_n^{-n}M) = \{\mathfrak{p} \in \text{Supp}(M) : depth(\mathfrak{p}, M) = depth_{R_\mathfrak{p}} M_\mathfrak{p} = n - 1\}$.

(4) $\mathrm{Ass}\,((U_f)_n^{-n}M) = \{\mathfrak{p} \in \mathrm{Supp}(M)\backslash\{\mathfrak{m}\} :$ for some $(a_1, \ldots, a_n)$ $\in (U_f)_n, (a_1, \ldots, a_n)R \subset \mathfrak{p}$ and $depth_{R_\mathfrak{p}} M_\mathfrak{p} = n - 1\}$.

DEFINITION 4. [Y, 1.2 and 1.3] We say that an R-module L is *cocyclic* if L is a submodule of $E(R/\mathfrak{m})$ for some $\mathfrak{m} \in \mathrm{Max}(R)$ where $E(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. In other words $L \subset \mathrm{Hom}(R, E(R/\mathfrak{m}))$.

Let M be an R-module. A prime ideal $\mathfrak{p}$ of R is called a *coassociated prime ideal* of M if there exists a cocyclic homomorphic image L of M such that $\mathfrak{p} = \mathrm{Ann}(L)$. The set of coassociated prime ideals of M is denoted by $\mathrm{Coass}(M)$.

DEFINITION 5. For an R-module M the subset $w(M)$ of R is defined by

$$w(M) = \{a \in R : M \xrightarrow{a\cdot} M \text{ is not surjective }\}.$$

REMARK 6. In [Y, 1.13], Yassemi introduced the dual version of $\bigcup_{\mathfrak{p}\in\mathrm{Ass}(M)} \mathfrak{p} = \{a \in R : M \xrightarrow{a\cdot} M$ is not injective $\}$, *i.e.*, he gave

$$w(M) = \bigcup_{\mathfrak{p}\in\mathrm{Coass}(M)} \mathfrak{p}.$$

The next Theorem shows the relationship between $w(U_n^{-n}M)$ and a property of the given triangular subset U_n.

THEOREM 7. *Let R be a Noetherian ring and $U_n^{-n}M$ a module of generalized fractions where M is an R-module. Then for each $a_1 \in R$ such that $(a_1, \ldots, a_n) \in U_n$, we have*

$$U_n^{-n}M = U_n^{-n}a_1M.$$

In particular, we have $a_1 \notin w(U_n^{-n}M)$.

PROOF. Let $\dfrac{s}{(b_1, \ldots, b_n)} \in U_n^{-n}M$. Then by the definition of generalized fractions there are $H, K \in D_n(R)$ and $(c_1, \ldots, c_n) \in U_n$ such that $H[a_1 \ \ldots \ a_n]^T = [c_1 \ \ldots \ c_n]^T = K[b_1 \ \ldots \ b_n]^T$. Hence we have $c_1 = h_{11}a_1$ for some $h_{11} \in R$. Therefore, by the definition of addition of generalized fractions, we get (for the first equality we may take $m = 0$ in below Definition 1, and for the second equality we can choose a suitable lower triangular matrix)

$$\frac{s}{(b_1, \ldots, b_n)} = \frac{|K|s}{(c_1, \ldots, c_n)} = \frac{c_1|K|s}{(c_1{}^2, c_2, \ldots, c_n)}$$
$$= \frac{a_1 h_{11}|K|s}{(c_1{}^2, c_2, \ldots, c_n)} \in U_n^{-n} a_1 M.$$

Next, since the multiplication by a_1 on $U_n^{-n}M \longrightarrow U_n^{-n}M$ is surjective by the above proof, we have the conclusion.

COROLLARY 8. *Let N be an R-module and $f : U_n^{-n}M \longrightarrow N$ be an R-homomorphism. Then we have, for each $a_1 \in R$ such that $(a_1, \ldots, a_n) \in U_n$,*

$$\text{if } f(U_n^{-n}a_1M) = 0, \ \text{ then } f(U_n^{-n}M) = 0.$$

PROOF. From the above Theorem 7 it is clear.

THEOREM 9. *Let R be a Noetherian ring and M a finitely generated R-module with $\dim M = d$. Then we have the following.*

(1) $\mathrm{Coass}(U_h)_n^{-n}M \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : ht_M\mathfrak{p} = 0\}$.
(2) $\mathrm{Coass}(U_s)_n^{-n}M \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : \dim M/\mathfrak{p}M = d\}$.
(3) $\mathrm{Coass}(U_r)_n^{-n}M \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : depth(\mathfrak{p}, M) = 0\}$.
(4) *If, in addition, R is local, then we have*
$\mathrm{Coass}(U_f)_n^{-n}M \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : depth_{R_\mathfrak{p}}M_\mathfrak{p} = 0\}$.

PROOF. (1) Let $\mathfrak{p} \in \mathrm{Coass}(U_h)_n^{-n}M$ and $ht_M\mathfrak{p} \geq 1$. Then by [Y, 1.7] we have

$$\mathfrak{p} \in \mathrm{Ass}\Big(\mathrm{Hom}\big((U_h)_n^{-n}M, E(R/\mathfrak{m})\big)\Big)$$

for some $\mathfrak{m} \in \mathrm{Max}(R)$. Hence there is $f \in \mathrm{Hom}((U_h)_n^{-n}M, E(R/\mathfrak{m}))$ such that $\mathrm{Ann}(f) = \mathfrak{p}$, that is, for the R-homomorphism $f : (U_h)_n^{-n}M \to E(R/\mathfrak{m})$ we have $\mathfrak{p}f((U_h)_n^{-n}M) = f((U_h)_n^{-n}\mathfrak{p}M) = 0$.

Then the following claim completes the proof.

CLAIM: $f((U_h)_n^{-n}M) = 0$. That is $f = 0$.

Proof of Claim. Assume that $f((U_h)_n^{-n}M) \neq 0$. Then there is $\dfrac{m}{(a_1,\dots,a_n)} \in (U_h)_n^{-n}M$ such that $f\left(\dfrac{m}{(a_1,\dots,a_n)}\right) \neq 0$.

Since $ht_M\mathfrak{p} \geq 1$, there is $b_1 \in \mathfrak{p}$ such that $(b_1, b_2, \dots, b_n) \in (U_h)_n$ by [M, 13.6]. By the definition of the modules of generalized fractions, there is $H, K \in D_n(R)$ and $(c_1,\dots,c_n) \in (U_h)_n$ such that $H[a_1 \ \dots \ a_n]^T = [c_1 \ \dots \ c_n]^T = K[b_1 \ \dots \ b_n]^T$. Hence we have $c_1 \in \mathfrak{p}$ and by the definition of addition of generalized fractions again

$$\frac{m}{(a_1,\dots,a_n)} = \frac{|H|m}{(c_1,\dots,c_n)} = \frac{c_1|H|m}{({c_1}^2, c_2,\dots,c_n)} \in (U_h)_n^{-n}\mathfrak{p}M.$$

Therefore we have

$$f\left(\frac{m}{(a_1,\dots,a_n)}\right) = f\left(\frac{c_1|H|m}{({c_1}^2, c_2,\dots,c_n)}\right) = 0.$$

This is a contradiction.

(2), (3) and (4) In the above proof, if we replace $ht_M\mathfrak{p} \geq 1$ by $dim\ M/\mathfrak{p}M < d$ (respectively, $depth(\mathfrak{p}, M) \geq 1$ and $depth_{R_\mathfrak{p}}M_\mathfrak{p} \geq 1$), then we have the same conclusions using the definitions of given triangular subsets and the similar method.

The next Theorem is an extended version of Sharp[S, 2.1] which was proved under the local ring. The proof follows his method.

THEOREM 10. *Let R be a ring, M an R-module and $\mathfrak{m}$ a maximal ideal of R. Suppose* $\mathrm{Ann}(x) \subset \mathfrak{m}$ *for all non-zero $x \in M$. Then we have*

(1) $f : M \longrightarrow \mathrm{Hom}(\mathrm{Hom}(M, E(R/\mathfrak{m})), E(R/\mathfrak{m}))$ *is injective.*

(2) $\mathrm{Ann}_R M = \mathrm{Ann}(\mathrm{Hom}(M, E(R/\mathfrak{m}))) = \mathrm{Ann}\Big(\mathrm{Hom}\big(\mathrm{Hom}(M, E(R/\mathfrak{m})), E(R/\mathfrak{m})\big)\Big)$.

PROOF. It is clear that

$$\begin{aligned}(0 :_R M) &\subset (0 :_R \mathrm{Hom}(M, E(R/\mathfrak{m})))\\ &\subset \big(0 :_R \mathrm{Hom}\big(\mathrm{Hom}(M, E(R/\mathfrak{m})), E(R/\mathfrak{m})\big)\big).\end{aligned}$$

Let $f : M \longrightarrow \mathrm{Hom}\big(\mathrm{Hom}(M, E(R/\mathfrak{m})), E(R/\mathfrak{m})\big)$ be the R-homo morphism for which $[f(x)]g = g(x)$ for all $g \in \mathrm{Hom}(M, E(R/\mathfrak{m}))$ and $x \in M$. Let $g' : Rx \longrightarrow E(R/\mathfrak{m})$ be the composition of the canonical maps

$$Rx \cong R/\mathrm{Ann}(x) \longrightarrow R/\mathfrak{m} \longrightarrow E(R/\mathfrak{m})$$

which is well-defined using $\mathrm{Ann}(x) \subset \mathfrak{m}$. Then we have $g' \neq 0$ and, since $E(R/\mathfrak{m})$ is injective, we can extend g' to $g : M \longrightarrow E(R/\mathfrak{m})$. Hence we obtain f is injective and

$$(0 :_R \mathrm{Hom}(\mathrm{Hom}(M, E(R/\mathfrak{m})), E(R/\mathfrak{m}))) \subset (0 :_R M).$$

These complete the proof.

REFERENCES

[C] S. C. Chung, *Associated prime ideals and isomorphisms of modules of generalized fractions*, Math. J. Toyama Univ., **17** (1994), 175–205.

[M] H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.

[S] R. Y. Sharp, *Some results on the vanishing of local cohomology modules*, Proc. London Math. Soc., **30** (1975), 177–195.

[SZ1] R. Y. Sharp and H. Zakeri, *Modules of generalized fractions*, Mathematika, **29** (1982), 32–41.

[SZ2] R. Y. Sharp and H. Zakeri, *Modules of generalized fractions and balanced big Cohen-Macaulay modules*, Commutative Algebra: Durham 1981, London Math. Soc. Lecture Note Ser., 72 (Cambridge University Press, 1982), 61–82.

[Y] S. Yassemi, *Coassociated primes*, Comm. Alg., **23(4)** (1995), 1473 - 1498.

DEPARTMENT OF MATHEMATICS
CHUNGNAM NATIONAL UNIVERSITY
TAEJON 305-764, KOREA