JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 9, July 1996

ON ISOMORPHISM THEOREMS IN BCI–SEMIGROUPS

SUN SHIN AHN* AND HEE SIK KIM**

ABSTRACT. In this paper, we consider the quotient algebra of BCI-semigroups, and obtain some isomorphism theorems of BCI-semigroups.

1. Introduction

In 1966, K. Iséki ([4]) introduced the notion of BCI-algebra which is a generalization of a BCK-algebra. A lot of works has been done on BCK-algebras and BCI-algebras. The ideal theory plays an important role in studying BCK-algebras and BCI-algebras, and some interesting results have been obtained ([2,3,5,6,10]). For BCK-algebras, every ideal is a subalgebra. Unfortunately, the ideal of BCI-algebra need not be a subalgebra. In 1993, Y.B. Jun et. al. ([7]) introduced the notion of BCI-semigroups/monoid, and studied their properties. They also considered the concept of $\mathcal{I}$-ideals in BCI-semigroups. The present authors ([1]) studied $\mathcal{I}$-ideals in BCI-semigroups, and some authors ([8,9]) studied BCI-semigroups related to the fuzzy theory. Since every p-semisimple BCI-algebra leads to an abelian group by defining $x+y := x*(0*y)$, p-semisimple BCI-semigroup gives a ring structure. On the while, every ring gives BCI-algebra by defining $x*y := x-y$ and hence leads to a BCI-semigroup. This means that the BCI-semigroup is a generalization of the ring structure. In

Supported (in part) by the BSRI Program, MOE, 1995, Project No. BSRI-95-1423.
Received by the editors on March 2, 1996.
1991 *Mathematics subject classifications*: Primary 06F35.

this paper, we consider the quotient algebras and obtain some isomorphism theorems of BCI-semigroups.

Recall that a *BCI-algebra* is a non-empty set X with a binary operation " $*$ " and a constant 0 satisfying the axioms:

BCI-1 $((x * y) * (x * z)) * (z * y) = 0$,

BCI-2 $(x * (x * y)) * y = 0$,

BCI-3 $x * x = 0$,

BCI-4 $x * y = 0$ and $y * x = 0$ imply that $x = y$,

BCI-5 $x * 0 = 0$ implies that $x = 0$,

for all $x, y, z \in X$. In a BCI-algebra X, we define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A non-empty subset A of a BCI-algebra X is said to be an *ideal* of X if (i) $0 \in A$, (ii) $x * y \in A$ and $y \in A$ imply that $x \in A$.

DEFINITION 1.1 ([7]). A *BCI-semigroup* is a non-empty set X with two binary operations " $*$ " and " $\cdot$ " and constant 0 satisfying the axioms:

(1) $(X; *, 0)$ is a BCI-algebra,

(2) $(X, \cdot)$ is a semigroup,

(3) the operation " $\cdot$ " is distributive (on both sides) over the operation " $*$ ", that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

We give some examples of the BCI-semigroup which is a generalization of the ring. We usually for convenience denote xy instead of $x \cdot y$ in a BCI-semigroup X.

EXAMPLE 1.2. Define two binary operations "$*$" and "$\cdot$" on a set $X := \{0, 1, 2, 3\}$ as follows:

$*$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	0	2	3

Then, by routine calculations, we can see that X is a BCI-semigroup.

EXAMPLE 1.3. Let $X := \{0, 1, 2, 3, 4\}$ be a set in which the operations "$*$" and "$\cdot$" are defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	2
3	0	0	0	2	3
4	0	1	2	3	4

Then $(X; *, \cdot, 0)$ is a BCI-semigroup.

PROPOSITION 1.4 ([7]). *Let X be a BCI-semigroup. Then*
(i) $0x = x0 = 0$,
(ii) $x \leq y$ *implies that* $xz \leq yz$ *and* $zx \leq zy$, *for all* $x, y, z \in X$.

DEFINITION 1.5 ([7]). A non-empty subset A of a BCI-semigroup X is called a *left(right) $\mathcal{I}$-ideal* of X if

(i) A is an ideal of a BCI-algebra X,
(ii) $a \in X$ and $x \in A$ imply that $ax \in A$ ($xa \in A$).

Both left and right $\mathcal{I}$-ideal is called a *two-sided $\mathcal{I}$-ideal* or simply *$\mathcal{I}$-ideal.* Let I be an $\mathcal{I}$-ideal of a BCI-algebra $(X; *, 0)$. For any x, y in X, we define $x \sim y$ by $x * y \in I$ and $y * x \in I$. Then $\sim$ is a congruence relation on X. Denote $X/I := \{C_x | x \in X\}$ and define that $C_x * C_y = C_{x*y}$. Since $\sim$ is a congruence relation on X, the operation "$*$" is well-defined.

THEOREM 1.6 ([10]). *Let* $(X;*,0)$ *be a* BCI*-algebra and let* I *be an ideal of* X. *Then* $(X/I;*,C_0)$ *is also a* BCI*-algebra, which is called the quotient algebra via* I, *and* $C_0 = I$.

DEFINITION 1.7. Let X and Y be BCI-semigroups. A map f is called a *BCI-semigroup homomorphism* if $f(x*y) = f(x)*f(y)$ and $f(x\cdot y) = f(x)\cdot f(y)$ for any $x, y \in X$.

2. Main Results

It is well known that in BCK-algebras every ideal is a subalgebra. But in BCI-algebras this fails to hold (see [10, p.498]). Hence in BCI-semigroups this also fails to hold. We say an ideal A of a BCI-algebra X *regular* in case A is also a subalgebra. Obviously, in BCK-algebras each ideal is regular.

For our purpose, we say an $\mathcal{I}$-ideal A of a BCI-semigroup X *regular* in case A is also a subalgebra of X.

DEFINITION 2.1. Let $(X;*,\cdot,0)$ be a BCI-semigroup and let X_0 be any non-empty subset of X. $(X_0;*,\cdot,0)$ is called a *subalgebra* of X if for any $x, y \in X_0$, $x*y \in X_0$ and $x\cdot y \in X_0$.

We can easily prove the following theorem and omit the proof.

THEOREM 2.2. *Let* X *and* Y *be* BCI*-semigroups, and let* $f: X \to Y$ *be a* BCI*-semigroup homomorphism. Then* $Ker f$ *is a regular* $\mathcal{I}$*-ideal of* X.

We define quotient algebra and obtain some isomorphisms of BCI-semigroups. Let I be an $\mathcal{I}$-ideal of a BCI-semigroup X. Then $(X/I;*,C_0)$ is a BCI-algebra by Theorem 1.6. Define $C_x \cdot C_y = C_{x\cdot y}$ on X/I. Then the operation " $\cdot$ " is well-defined. If $C_x = C_y$ and $C_s = C_t$, then $x*y, y*x \in I$ and $s*t, t*x \in I$. Since I is an $\mathcal{I}$-ideal of a BCI-semigroup X, $y(t*s) = yt*ys \in I$ for any $y \in X$, and $(y*x)s = ys*xs \in I$ for any $s \in X$. Then,

$(yt * xs) * (ys * xs) \leq yt * ys \in I$ and so $(yt * xs) * (ys * xs) \in I$. Since $ys * xs \in I$ and I is an $\mathcal{I}$-ideal of X, $yt * xs \in I$. Similarly, $xs * yt \in I$. Since $xs * yt \in I$ and $xs * yt \in I$, $C_{xs} = C_{yt}$. Therefore $C_x \cdot C_s = C_y \cdot C_t$. Hence " $\cdot$ " is well-defined.

We claim that $(X/I, \cdot)$ is a semigroup. Since $C_x \cdot (C_y \cdot C_z) = C_x \cdot (C_y \cdot C_z) = C_{x(yz)} = C_{(xy)z} = (C_{xy}) \cdot C_z = (C_x \cdot C_y) \cdot C_z$ for any $C_x, C_y, C_z \in X/I$, $(X/I, \cdot)$ is a semigroup.

Finally, we show that "$\cdot$" is distributive (on both sides) over the operation "$*$". For any $C_x, C_y, C_z \in X/I$, $C_x \cdot (C_y * C_z) = C_x \cdot (C_{y*z}) = C_{x(y*z)} = C_{xy*xz} = C_{xy} * C_{xz} = (C_x \cdot C_y) * (C_x \cdot C_z)$. By a similar way, we obtain $(C_x * C_y) \cdot C_z = (C_x \cdot C_z) * (C_y \cdot C_z)$. Hence $(X/I; *, \cdot, C_0)$ satisfies distributive laws. Thus X/I is a BCI-semigroup. Summarizing the above facts we have the following theorem:

THEOREM 2.3. *Let $(X; *, \cdot, 0)$ be a BCI-semigroup and let I be an $\mathcal{I}$-ideal of X. Then $(X/I; *, \cdot, C_0)$ is also a BCI-semigroup which is called the quotient algebras via I, and $C_0 = I$.*

Furthermore, if 1_X is the multiplicative identity of a BCI-semigroup X, then C_{1_X} is the multiplicative identity of X/I ; and if X is commutative, then X/I is also commutative. If $I = X$, then X/I is the zero BCI-semigroup. If $I = (0)$, then X/I is the same as the BCI-semigroup X by identifying C_a, $a \in X$, with a.

THEOREM 2.4. *Let X and Y be BCI-semigroups, and let $f : X \rightarrow Y$ be a BCI-semigroup epimorphism. Then $Y \cong X/Kerf$.*

PROOF. By Theorem 2.2, $Kerf$ is a regular $\mathcal{I}$-ideal of X. So the quotient algebra $X/Kerf$ can be obtained in the standard way. Now we define $h : Y \rightarrow X/Kerf$ by $h(f(x) = (Kerf)_x$. We will prove that h is an isomorphism from Y onto $X/Kerf$. First, we claim that h is well-defined. If $f(x) = f(y)$, then $0 = f(x) * f(y) = f(y) * f(x)$

and so $f(x*y) = f(y*x) = 0$. It follows that $x*y,\ y*x \in Kerf$ and hence $(Kerf)_x = (Kerf)_y$. On the other hand, for each $y \in Y$, since f is an epimorphism, there exists an element x in X such that $f(x) = y$. So $h(y) = h(f(x)) = (Kerf)_x$. Since $h(f(x)*f(y)) = h(f(x*y)) = (Kerf)_{x*y} = (Kerf)_x * (Kerf)_y = h(f(x)) * h(f(y))$ and $h(f(x)f(y)) = h(f(xy)) = (Kerf)_{xy} = (Kerf)_x \cdot (Kerf)_y = h(f(x)) \cdot h(f(y))$, h is a BCI-semigroup homomorphism.

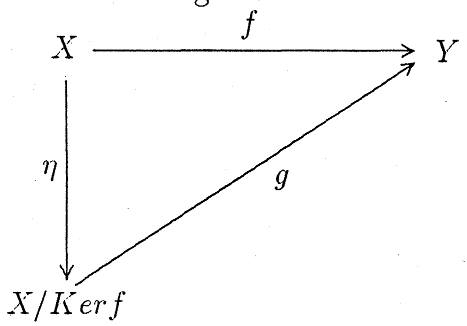
Suppose that $h(y) = (Kerf)_0$ for some y in Y. Then there exists $x \in X$ such that $f(x) = y$. So $(Kerf)_0 = h(f(x)) = (Kerf)_x$. This means $x = x*0 \in Kerf$ and $y = f(x) = 0$, therefore h is one-one.

For each $(Kerf)_x \in X/Kerf$, we have $h(f(x)) = (Kerf)_x$. Hence h is onto. This completes the proof.

COROLLARY 2.5. *Let I be an $\mathcal{I}$-ideal of a BCI-semigroup X, and let $\eta : X \to X/I$ be the natural BCI-semigroup homomorphism. Then $X/I \cong X/Ker\eta$.*

The proof can be immediately obtained from Theorem 2.4. In the above Corollary 2.5, $Ker\eta = I_0$, where I_0 is the class containing 0 in the quotient algebra X/I. In general, $I \supseteq Ker\eta = I_0$.

THEOREM 2.6. *Given a BCI-semigroup homomorphism $f : X \to Y$, there exists a unique injective BCI-semigroup homomorphism $g : X/Kerf \to Y$ such that the diagram*



commutes, that is, $f = g\eta$, where η is the canonical (or natural)

BCI-semigroup homomorphism.

PROOF. Clearly, the map g defined by $g((Kerf)_x) = f(x)$ is injective. Also, $f = g\eta$, as proved in Theorem 2.4. To show that g is unique, let $f = h\eta$, where $h : X/Kerf \to Y$ is a BCI-semigroup homomorphism. Then $g\eta(x) = h\eta(x)$ for all $x \in X$. So $g((Kerf)_x) = h((Kerf)_x)$. This proves $g = h$.

DEFINITION 2.7. Let X and Y be BCI-semigroups. A BCI-homomorphism $f : X \to Y$ is said to be *regular* if Imf is an $\mathcal{I}$-ideal of Y.

By the definition, for any subalgebra A of X, A is a regular $\mathcal{I}$-ideal if and only if the inclusion mapping $\iota : A \to X$ is a regular BCI-semigroup homomorphism.

PROPOSITON 2.8. *Let $f : X \to Y$ be a regular BCI-semigroup homomorphism. Then for each $\mathcal{I}$-ideal I with $I \supseteq Kerf$, $f(I)$ is an $\mathcal{I}$ -ideal of $f(X)$. Conversely, for each $\mathcal{I}$ -ideal K of Y, $f^{-1}(K) := \{a \in X | f(a) \in K\}$ is an $\mathcal{I}$-ideal of X.*

PROOF. Assume $y * b, b \in f(I) \subseteq f(X)$. Then there exists $a \in I$ such that $b = f(a)$. Since $f(x)$ is an $\mathcal{I}$-ideal of Y, $y \in f(X)$. So there exists $x \in X$ such that $f(x) = y$. It follows $f(x) * f(a) = f(c)$ for some c in I. Thus $f((x * a) * c) = 0$ and so $(x * a) * c \in Kerf \subseteq I$. Since I is an $\mathcal{I}$-ideal of X, $x * a \in I$, it follows $x \in I$. Therefore $y = f(x) \in f(I)$. Clearly $0 \in f(I)$. Suppose $b \in f(I)$ and $y \in f(X)$. Then there exist $x, a \in X$ such that $f(x) = y$ and $f(a) = b$. It follows that $by = f(a)f(x) = f(ax) \in f(I)$ and $yb = f(xb) \in f(I)$. Hence $f(I)$ is an $\mathcal{I}$-ideal of $f(X)$.

Conversely, let $x * a$, $a \in f^{-1}(K)$. Then $f(x) * f(a), f(a) \in K$. Since K is an $\mathcal{I}$-ideal of Y, $f(x) \in K$. It follows $x \in f^{-1}(K)$. If $x_1, x_2 \in f^{-1}(K)$, then $f(x_1), f(x_2) \in K$. Since K is an $\mathcal{I}$-ideal of Y,

$f(x_1x_2) = f(x_1)f(x_2) \in K$. It follows $x_1x_2 \in f^{-1}(K)$. Obviously $0 \in f^{-1}(K)$. Therefore, $f^{-1}(K)$ is an $\mathcal{I}$-ideal of X.

Let $\eta : X \to X/A$ be the natural BCI-semigroup homomorphism of X onto X/A. By applying Proposition 2.8, we have the following theorem:

THEOREM 2.9. *Let X be a BCI-semigroup and A be a regular $\mathcal{I}$-ideal of X. Then there is an one-to-one correspondence between the $\mathcal{I}$-ideals of X containing A and the $\mathcal{I}$-ideals of X/A.*

REFERENCES

1. S. S. Ahn and H. S. Kim, *A note on $\mathcal{I}$-ideals in BCI-semigroups*, Comm. Korean Math. Soc. (to appear).
2. J. Ahsan, A.B. Thaheen, *On ideals in BCK-algebras*, Math. Seminar Notes **5** (1977), 167-172.
3. E.Y. Deeba, *On the algebra of ideals of a BCK-algebra*, Math. Japonica **30** (1985), 391-393.
4. K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan. Acad. **42** (1966), 351-366.
5. K. Iséki, *On ideals in BCK-algebra*, Math. Seminar Notes **3** (1975), 1-12.
6. K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japonica **21** (1976), 351-366.
7. Y.B. Jun, S.M. Hong and E.H. Roh, *BCI-semigroups*, Honam. Math. J. **15** (1993), 59-64.
8. Y. B. Jun, S. S. Ahn, J. Y. Kim and H. S. Kim, *Fuzzy $\mathcal{I}$-ideals in BCI-semigroups*, submitted.
9. Y. B. Jun, Y. H. Kim, J. Y. Kim and H. S. Kim, *Fuzzy commutative $\mathcal{I}$-ideals in BCI-semigroups*, submitted.
10. C.Z. Mu and W.H. Xiong, *On ideals in BCK-algebras*, Math. Japonica **36** (1991), 497-501.

*

Department of Mathematics Education
Dongguk University
Seoul 100-715, Korea
**

Department of Mathematics Education
Chungbuk National University
Chongju 361-763, Korea