An Examination on the Singularity of Grad Moment Equations for Shock Wave Problems

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It has been well known that the Grad thirteen-moment equations have solutions only when the Mach number is less than a limiting value for the stationary plane shock-waves. The limit of Mach number has been re-examined by including successive terms in the series expansion of distribution function. The method employed is the linear analysis of moment equations near up-streaming and down-streaming flows. For the thirteen moment case, it has been confirmed that equations have solutions only when the Mach number is less than 1.6503, which is consistent with the literature value. For the case of twenty moments, the limit of Mach number is decreased to 1.3416.

Introduction

The shock wave problem is one of the classical problems in the development of kinetic theory of gases.[1] It has been understood from several different theories.[2] For weak plane shock-waves, the Navier-Stokes theory predicts a smooth shock profile which is very close to that observed experimentally.[2] The shock-strength is characterized by the Mach number M of shock speed.

For strong shocks, more realistic results are obtained from the Boltzmann equation.[2] However, it has turned out that classical methods of solution have inherent deficiencies for strong shock waves.[3] For example, Burnet equations have no solutions for $M>2.1$ and Grad 13-moment equations for $M>1.65$. After the note of Holway,[4] it has been recognized that the mathematical reason of deficiency in the Grad moment equations is that the series expansion employed in the method does not converge above certain critical Mach number.

In the Grad moment method,[5] the velocity distribution function of gases is expanded in the form

$$f(\boldsymbol{v},\boldsymbol{r},t)=f^{(0)}\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}\Theta^{(\alpha)}(\boldsymbol{r},t)\circledast H^{(\alpha)}(\boldsymbol{w}) \tag{1}$$

where $f^{(0)}$ is the local Maxwellian; $H^{(\alpha)}(\boldsymbol{w})$ is the tensor Hermite polynomial of the α-th order with $\boldsymbol{w}=(m/k_BT)^{1/2}(\boldsymbol{v}-\boldsymbol{u})$, the local fluid velocity $\boldsymbol{u}$, the molecular mass m, the local temperature T, and Boltzmann constant k_B; $\Theta^{(\alpha)}$ is the tensor expansion coefficient which is the moment of $H^{(\alpha)}$; and the symbol $\circledast$ represents the scalar product between two tensors. In the 13-moment approximation, the expansion coefficients beyond the first 13 are truncated. When the additional terms are retained in the expansion, for example, taking the first 20 moments, the accuracy of the distribution function should be increased provided that the series expansion converges. Holway stated in his note that limit of convergence will asymptotically approach $M=1.851$ as the number of terms in the expansion is increased.[4,8]

In this paper, we re-examine the critical Mach number of Grad moment equations by including successive terms in the series expansion. We will not try to show rigorous mathematical reasons for the presence of singularity in the moment equations. The purpose of present work is to show that the limit of Mach number does not increase as contrasted with Holway's statement, when more terms are added in the expansion.

Formulation of the Shock Wave Problem

Let the x axis be directed along the flow perpendicular to a shock wave. The shock itself is travelling in the negative x direction with constant speed. However, it can be arranged to locate the shock layer to a fixed position by performing Galilei transformation. In this coordinate frame, the gases far from the shock layer constitute the up-streaming flow at $x\rightarrow-\infty$ with the speed u' which is equal to the shock speed in opposite direction, and the down-streaming flow at $x\rightarrow+\infty$ with a speed u^f. The gases far from shock layer are in equilibrium so that the corresponding distribution functions are Maxwellian.

For the one-dimensional problem under consideration, the Boltzmann equation takes the form

$$v_x\frac{\partial f}{\partial x}=\int gI(g,\ \Omega)(f'f'_1-ff_1)d^2\Omega d^3v_1 \tag{2}$$

with conventional notations.[6] The mass, momentum and energy conservation equations are followed from (2):

$$\frac{d}{dx}(\rho u)=0 \tag{3a}$$

$$\frac{d}{dx}(P_{xx}+\rho u^2)=0 \tag{3b}$$

$$\frac{d}{dx}\left(Q_x+P_{xx}u+\rho\mathcal{E}u+\frac{1}{2}\rho u^3\right)=0 \tag{3c}$$

where the mass density, ρ; the x-component of flow velocity, u; the xx-component of pressure tensor, P_{xx}; the x-component of heat flux, Q_x; and the internal energy density, $\mathcal{E}$ are defined as follows.

$$\rho=\int mf(x,\boldsymbol{v})d^3\boldsymbol{v} \tag{4a}$$

$$\rho u=\int mv_xf(x,\boldsymbol{v})d^3\boldsymbol{v} \tag{4b}$$

$$P_{xx}=\int mC_x^2f(x,\boldsymbol{v})d^3\boldsymbol{v} \tag{4c}$$

$$Q_x=\int\frac{1}{2}mC_xC^2f(x,v)d^3v \quad (4d)$$

$$\rho\mathcal{E}=\int\frac{1}{2}mC^2f(x,v)d^3v \quad (4e)$$

where $C_x=v_x-u$ and $C^2=(v_x-u)^2+v_y^2+v_z^2$. With the conventional definition of kinetic temperature, the internal energy density is related to the temperature for dilute gases as $\mathcal{E}=3k_BT/2m$. At equilibrium, the P_{xx} reduces to $\rho k_BT/m$ and the Q_x vanishes.

Integrating the both sides of (3a-c) from $-\infty$ to $+\infty$, the well-known Rankine-Hugoniot conditions are derived.

$$\rho^iu^i=\rho^fu^f \quad (5a)$$

$$\rho^ik_BT^i/m+\rho^i(u^i)^2=\rho^fk_BT^f/m+\rho^f(u^f)^2 \quad (5b)$$

$$5\rho^iu^ik_BT^i/m+\rho^i(u^i)^3=5\rho^fu^fk_BT^f/m+\rho^f(u^f)^3 \quad (5c)$$

where the superscripts i and f denote the values at the up-stream and the down-stream, respectively. It is convenient to define dimensionless quantities,

$$\tilde{\rho}_0=\rho^f/\rho^i,\ \tilde{u}_0=u^f/u^i,\ \tilde{T}_0=T^f/T^i \quad (6)$$

and introduce a parameter $B\equiv(3/5)M^{-2}$ where the M is the Mach number of up-streaming flow given by

$$M=u^i(3m/5k_BT^i)^{1/2} \quad (7)$$

In general, the Mach number can take any real number of $1<M<\infty$, so that $0.6>B>0$. The Rankine-Hugoniot conditions are rewritten by

$$\tilde{\rho}_0\tilde{u}_0=1 \quad (8a)$$

$$\tilde{u}_0+B\tilde{\rho}_0\tilde{T}_0=1+B \quad (8b)$$

$$\tilde{u}_0^2+5B\tilde{T}_0=1+5B \quad (8c)$$

which constitute boundary conditions of the shock profile. The two sets of solutions of (8a-c)

$$\tilde{\rho}_0^i=1,\ \tilde{u}_0^i=1,\ \tilde{T}_0^i=1 \quad (9a)$$

and

$$\tilde{\rho}_0^f=\frac{4}{1+5B},\ \tilde{u}_0^f=\frac{1+5B}{4},\ \tilde{T}_0^f=\frac{(3-B)(1+5B)}{16B} \quad (9b)$$

are limiting values of the up-stream and the down-stream, respectively.

The conservation equations of shock wave are also obtained by integrating (3a-c) from $-\infty$ to a position in the shock zone,

$$\rho^iu^i=\rho u \quad (10a)$$

$$\rho^ik_BT^i/m+\rho^i(u^i)^2=P_{xx}+\rho u^2 \quad (10b)$$

$$5\rho^iu^ik_BT^i/m+\rho^i(u^i)^3=2uP_{xx}+2Q_x+3\rho uk_BT/m+\rho u^3 \quad (10c)$$

Defining the dimensionless quantities

$$\tilde{\rho}=\rho/\rho^i,\ \tilde{u}=u/u^i,\ \tilde{T}=T/T^i \quad (11a)$$

$$\tilde{P}=P_{xx}/[\rho^i(u^i)^2],\ \tilde{Q}=Q_x/[\rho^i(u^i)^3]. \quad (11b)$$

(10a-c) can be rewritten by

$$\widetilde{\rho u}=1 \quad (12a)$$

$$\tilde{P}+\tilde{u}=1+B \quad (12b)$$

$$2\tilde{u}\tilde{P}+2\tilde{Q}+3B\tilde{T}+\tilde{u}^2=1+5B \quad (12c)$$

The Navier-Stokes theory employs the linear constitutive relations for unknown $\tilde{P}$ and $\tilde{Q}$. In the moment method, one explores the equations of $\tilde{P}$ and $\tilde{Q}$ which are described in the following section.

Moment Equations

The derivations of moment equations are straightforward. For P_{xx}, multiplying both sides of (2) by mv_x^2 and performing integration, one gets

$$\frac{d}{dx}(R^{(P)}+3uP_{xx}+\rho u^3)=\Lambda^{(P)} \quad (13)$$

where

$$R^{(P)}=\int mC_x^3f(x,\ v)d^3v \quad (14a)$$

$$\Lambda^{(P)}=\int\int mC_x^2gI(f'_1f'-f_1f)d^2\Omega d^3v_1d^3v. \quad (14b)$$

The equation of Q_x is obtained with the same way multiplying by mv_xv^2. After some rearrangements, one gets

$$\frac{d}{dx}\left(R^{(Q)}+4uQ_x-u^2P_{xx}+\frac{3k_BT}{m}\rho u^2-\rho u^4\right)$$
$$+2(R^{(P)}+3uP_{xx}+\rho u^3)\frac{du}{dx}=\Lambda^{(Q)} \quad (15)$$

where

$$R^{(Q)}=\int mC_x^2C^2f(x,\ v)d^3v \quad (16a)$$

$$\Lambda^{(Q)}=\int\int mC_xC^2gI(f'_1f'-f_1f)d^2\Omega d^3v_1d^3v. \quad (16b)$$

The equations for higher moments $R^{(P)}$ and $R^{(Q)}$ are also obtained similarly. The additional moment equation examined in the present work is

$$\frac{d}{dx}(R^{(H)}+uR^{(P)}-3u^2P_{xx}-2\rho u^4)$$
$$+3(R^{(P)}+3uP_{xx}+\rho u^3)\frac{du}{dx}=\Lambda^{(H)} \quad (17)$$

where

$$R^{(H)}=\int mC_x^4f(x,\ v)d^3v \quad (18a)$$

$$\Lambda^{(H)}=\int\int mC_x^3gI(f_1'f'-f_1f)d^2\Omega d^3v_1d^3v \quad (18b)$$

In order to simplify equations, following reduced quantities are introduced.

$$\tilde{R}^{(P)}=R^{(P)}/[\rho^i(u^i)^3],\ \tilde{\Lambda}^{(P)}=\lambda\Lambda^{(P)}/[\rho^i(u^i)^3] \quad (19a)$$

$$\tilde{R}^{(Q)}=R^{(Q)}/[\rho^i(u^i)^4],\ \tilde{\Lambda}^{(Q)}=\lambda\Lambda^{(P)}/[\rho^i(u^i)^4] \quad (19b)$$

$$\tilde{R}^{(H)}=R^{(H)}/[\rho^i(u^i)^4],\ \tilde{\Lambda}^{(H)}=\lambda\Lambda^{(H)}/[\rho^i(u^i)^4] \quad (19c)$$

and $\tilde{x}=x/\lambda$, where λ is an appropriate length parameter which one may put the mean free path of up-streaming molecules. The moment equations are rewritten in terms of reduced quantities as follows.

$$(1+B-4\tilde{P})\frac{d\tilde{P}}{d\tilde{x}}+\frac{d\tilde{R}^{(P)}}{d\tilde{x}}=\tilde{\Lambda}^{(P)} \quad (20a)$$

$$-[B(3-B)+2(1+B)\tilde{P}-\tilde{P}^2+2\tilde{Q}+2\tilde{R}^{(P)}]\frac{d\tilde{P}}{d\tilde{x}}$$
$$+2(1+B-\tilde{P})\frac{d\tilde{Q}}{d\tilde{x}}+\frac{d\tilde{R}^{(Q)}}{d\tilde{x}}=\tilde{\Lambda}^{(Q)} \quad (20b)$$

$$-[3(1+B-\tilde{P})\tilde{P}+4\tilde{R}^{(P)}]\frac{d\tilde{P}}{d\tilde{x}}$$
$$+(1+B-\tilde{P})\frac{d\tilde{R}^{(P)}}{d\tilde{x}}+\frac{d\tilde{R}^{(H)}}{d\tilde{x}}=\tilde{\Lambda}^{(H)} \quad (20c)$$

For writing (20a-c), the quantities $\tilde{\rho}$, $\tilde{u}$ and $\tilde{T}$ are eliminated using the relations (12a-c).

Singularity in the Thirteen Moment Equations

In the thirteen moment approximation, there are thirteen variables in the general three dimensional case.[5] In one dimension which is the case of present study, there are only five, namely ρ, u, T, P_{xx} and Q_x with the notations used in this paper. The distribution function is approximated by the form

$$f=f^{(0)}[1+\frac{1}{4}\Theta_{xx}^{(2)}(3w_x^2-w^2)+\frac{1}{5}\theta_x^{(3)}w_x(w^2-5)] \quad (21)$$

where

$$\Theta_{xx}^{(2)}=\frac{1}{\rho}(m/k_BT)P_{xx}-1 \quad (22a)$$

$$\theta_x^{(3)}=\frac{1}{\rho}(m/k_BT)^{3/2}Q_x \quad (22b)$$

and

$$w_x=(m/k_BT)^{1/2}C_x \quad (23a)$$

$$w^2=(m/k_BT)C^2. \quad (23b)$$

The use of (21) expresses the higher order moments in terms of the five quantities, for example,

$$R^{(P)}=\frac{6}{5}Q_x \quad (24a)$$

$$R^{(Q)}=(7P_{xx}-2\rho k_BT/m)k_BT/m. \quad (24b)$$

Substituting the (24a, b) into the moment equations (13) and (15), one obtains two differential equations for unknown P_{xx} and Q_x. With the notation of dimensionless quantities, they are written in the form

$$D_{11}\frac{d\tilde{P}}{d\tilde{x}}+D_{12}\frac{d\tilde{Q}}{d\tilde{x}}=\tilde{\Lambda}^{(P)} \quad (25)$$

$$D_{21}\frac{d\tilde{P}}{d\tilde{x}}+D_{22}\frac{d\tilde{Q}}{d\tilde{x}}=\tilde{\Lambda}^{(Q)} \quad (26)$$

where

$$D_{11}=1+B-4\tilde{P} \quad (27a)$$

$$D_{12}=\frac{6}{5} \quad (27b)$$

$$D_{21}=\frac{4}{3}B(3-B)-2(1+B)\tilde{P}+8\tilde{P}^2-\frac{136}{15}\tilde{Q}$$
$$-\frac{8}{9}\tilde{P}[B(3-B)+\tilde{P}^2-2\tilde{Q}]/(1+B-\tilde{P})$$
$$-\frac{2}{9}[B(3-B)+\tilde{P}^2-2\tilde{Q}]^2/(1+B-\tilde{P})^2 \quad (27c)$$

$$D_{22}=2\left(1+B-\frac{10}{3}\tilde{P}\right)$$
$$+\frac{8}{9}[B(3-B)+\tilde{P}^2-2\tilde{Q}]/(1+B-\tilde{P}) \quad (27d)$$

The equations (25, 26) have solutions only when

$$D_{11}D_{22}-D_{12}D_{21}\neq 0. \quad (28)$$

In order to look for the condition that the (28) does not hold, it is recommended to linearize the equations (25, 26) near the up-streaming ($x\to-\infty$), and the down-streaming ($x\to+\infty$) limits. From the (9a, b) and the (12a-c), ones see that the limiting values of $\tilde{P}$ and $\tilde{Q}$ are

$$\tilde{P}(x\to-\infty)=B \quad (29a)$$

$$\tilde{P}(x\to+\infty)=\frac{1}{4}(3-B) \quad (29b)$$

$$\tilde{Q}(x\to\pm\infty)=0 \quad (29c)$$

and the collision integrals $\tilde{\Lambda}^{(P)}$ and $\tilde{\Lambda}^{(Q)}$ vanish at equilibrium. Let us consider following expansions with an order parameter ε

$$\tilde{P}=\tilde{P}^{(0)}+\varepsilon\tilde{P}^{(1)}+\cdots \quad (30a)$$

$$\tilde{Q}=\varepsilon\tilde{Q}^{(1)}+\cdots \quad (30b)$$

$$\tilde{\Lambda}^{(P)}=\varepsilon\tilde{\Lambda}_P^{(1)}+\cdots \quad (30c)$$

$$\tilde{\Lambda}^{(Q)}=\varepsilon\tilde{\Lambda}_Q^{(1)}+\cdots \quad (30d)$$

where $\tilde{P}^{(0)}=B$ for the up-streaming and $(3-B)/4$ for the down-streaming. The differential equations (25, 26) are linearized by taking linear terms with respect to the parameter ε

$$D_{11}^{(0)}\frac{d\tilde{P}^{(1)}}{d\tilde{x}}+D_{12}^{(0)}\frac{d\tilde{Q}^{(1)}}{d\tilde{x}}=\tilde{\Lambda}_P^{(1)} \quad (31a)$$

$$D_{21}^{(0)}\frac{d\tilde{P}^{(1)}}{d\tilde{x}}+D_{22}^{(0)}\frac{d\tilde{Q}^{(1)}}{d\tilde{x}}=\tilde{\Lambda}_Q^{(1)} \quad (31b)$$

where

$$D_{11}^{(0)}=1-3B \quad (32a)$$

$$D_{12}^{(0)}=\frac{6}{5} \quad (32b)$$

$$D_{21}^{(0)}=2B \quad (32c)$$

$$D_{22}^{(0)}=2(1-B) \quad (32d)$$

near the up-streaming flow, and

$$D_{11}^{(0)}=2(B-1) \quad (33a)$$

$$D_{12}^{(0)}=\frac{6}{5} \quad (33b)$$

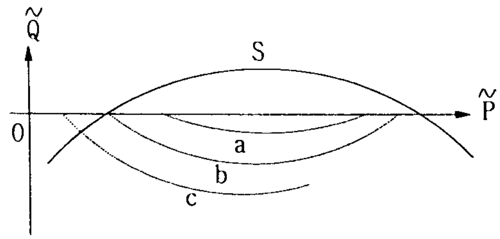

Figure 1. A shematic diagram of solutions of 13-moment equations in the phase space. The curve S represents a singular line given by $D_{11}D_{22}-D_{12}D_{21}=0$, and the dotted curves are hypothetical loci of solution points.

$$D_{21}^{(0)}=\frac{1}{8}(3+14B-5B^2) \tag{33c}$$

$$D_{22}^{(0)}=3B-1 \tag{33d}$$

near the down-streaming flow. The linearized equations (31a, b) do not have solutions when

$$D_{11}^{(0)}D_{22}^{(0)}-D_{12}^{(0)}D_{21}^{(0)}=0. \tag{34}$$

Inserting the (32a-d) and (33a-d) into the (34), ones obtain the equations of B at which the linear differential equations break down,

$$15B^2-26B+5=0 \tag{35a}$$

and

$$135B^2-202B+31=0 \tag{35b}$$

near the up-streaming and the down-streaming, respectively. Let us remind that the B is related to the Mach number by $M=(5B/3)^{-1/2}$ and $0.6>B>0$. The acceptable solutions are $B=0.2203$ (or $M=1.6503$) from (35a), and $B=0.1736$ (or $M=1.8591$) from (35b). Therefore, the lowest Mach number is 1.6503 for the linear equations to be unsolvable. This is the same value which Grad obtained in 1952.[7]

The lowest Mach number obtained by the linear equations is the limiting number for the original equations (25, 26) to be unsolvable. That is, the equations (25, 26) have solutions only when the Mach number is less than the limiting number 1.6503. This fact is illustrated schematically in the Figure 1. The Figure shows the hypothetical loci of solutions of (25, 26) in phase space with dotted curves for three cases of $M<M_0$ (curve a), $M=M_0$ (curve b) and $M>M_0$ (curve c) where $M_0=1.6503$. The solid curve (curve S) represents the singular line given by $D_{11}D_{22}-D_{12}D_{21}=0$. When $M=M_0$, the phase point of up-streaming flow coincides with the singular line. For $M>M_0$, the up-streaming phase point crosses over the line, so that the phase points of up-streaming and down-streaming are located at opposite sites over the singular line. Therefore, it is impossible to have a locus connecting the two phase points without coinciding with the singular line.

Singularity in the Twenty Moment Equations

In the twenty moment approximation, the higher order moments $\Theta^{(a)}$ above the third order are truncated in the infinite series of (1). The distribution function is expressed in terms of the first twenty moments in the general three-dimensional case.[5] In the present one-dimensional case, there are six variables, namely ρ u, T, P_{xx}, Q_x and $R^{(P)}$. The (1) is written in the form

$$f=f^{(0)}\Big[1+\frac{1}{4}\Theta_{xx}^{(2)}(3w_x^2-w^2)+\frac{1}{2}\theta_x^{(3)}(w_xw^2-w_x^3-2w_x)+\frac{1}{12}\Theta_{xxx}^{(3)}(5w_x^3-3w_xw^2)\Big] \tag{36}$$

where

$$\Theta_{xxx}^{(3)}=\frac{1}{\rho}(m/k_BT)^{3/2}R^{(P)} \tag{37}$$

and the $\Theta_{xx}^{(2)}$ and $\theta_x^{(3)}$ are defined in the previous section. Using the (36), one calculates the $R^{(Q)}$ and $R^{(H)}$ as follows

$$R^{(Q)}=7(k_BT/m)P_{xx}-2\rho(k_BT/m)^2 \tag{38a}$$

$$R^{(H)}=6(k_BT/m)P_{xx}-3\rho(k_BT/m)^2 \tag{38b}$$

or

$$\tilde{R}^{(Q)}=7B\tilde{T}\tilde{P}-2B^2\tilde{\rho}\tilde{T}^2 \tag{39a}$$

$$\tilde{R}^{(H)}=6B\tilde{T}\tilde{P}-3B^2\tilde{\rho}\tilde{T}^2 \tag{39b}$$

Substituting (39a, b) into the moment equations (20a-c) with the use of (12a-c), one obtains a set of differential equations for $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}^{(P)}$ in the form

$$\begin{bmatrix} E_{11} & 0 & 1 \\ E_{21} & E_{22} & 0 \\ E_{31} & E_{32} & E_{33} \end{bmatrix}\begin{bmatrix} d\tilde{P}/d\tilde{x} \\ d\tilde{Q}/d\tilde{x} \\ d\tilde{R}^{(P)}/d\tilde{x} \end{bmatrix}=\begin{bmatrix} \tilde{\Lambda}^{(P)} \\ \tilde{\Lambda}^{(Q)} \\ \tilde{\Lambda}^{(H)} \end{bmatrix} \tag{40}$$

where the matrix elements E_{ij}'s are defined by

$$E_{11}=1+B-4\tilde{P} \tag{41a}$$

$$E_{21}=\frac{4}{3}B(3-B)-2(1+B)\tilde{P}+8\tilde{P}^2-\frac{20}{3}\tilde{Q}-2\tilde{R}^{(P)}-\frac{8}{9}\tilde{P}[B(3-B)+\tilde{P}^2-2\tilde{Q}]/(1+B-\tilde{P})-\frac{2}{9}[B(3-B)+\tilde{P}^2-2\tilde{Q}]^2/(1+B-\tilde{P})^2 \tag{41b}$$

$$E_{22}=2\left(1+B-\frac{20}{3}\tilde{P}\right)+\frac{8}{9}[B(3-B)+\tilde{P}^2-2\tilde{Q}]/(1+B-\tilde{P}) \tag{41c}$$

$$E_{31}=2B(3-B)-3(1+B)\tilde{P}+9\tilde{P}^2-4\tilde{Q}-4\tilde{R}^{(P)}-\frac{4}{3}\tilde{P}[B(3-B)+\tilde{P}^2-2\tilde{Q}]/(1+B-\tilde{P})-\frac{1}{3}[B(3-B)+\tilde{P}^2-2\tilde{Q}]^2/(1+B-\tilde{P})^2 \tag{41d}$$

$$E_{32}=-4\tilde{P}+\frac{4}{3}[B(3-B)+\tilde{P}^2-2\tilde{Q}]/(1+B-\tilde{P}) \tag{41e}$$

$$E_{33}=1+B-\tilde{P} \tag{41f}$$

The (40) has solutions only when the determinant of matrix $[E_{ij}]$ does not vanish.

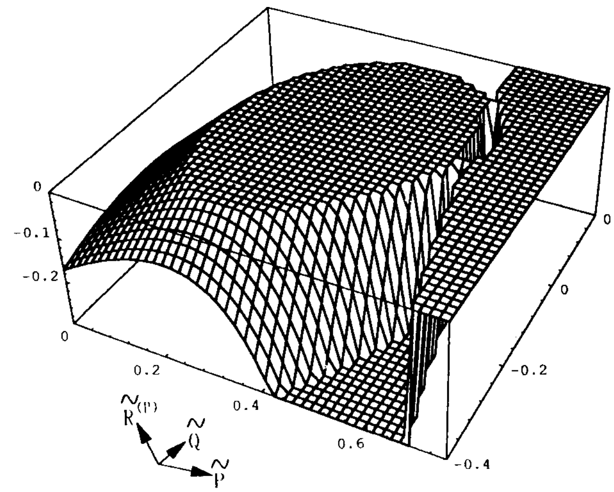

Figure 2. The singular surface given by $\mathcal{D}=0$ in the 20-moment case at $B=0.2$.

$$\mathcal{D}\equiv E_{11}E_{22}E_{33}+E_{21}E_{32}-E_{22}E_{31}\neq 0 \quad (42)$$

The same procedure as the previous section can be applied in this case. Since the limiting value of $R^{(P)}$ is

$$R^{(P)}(x\rightarrow\pm\infty)=0 \quad (43)$$

the differential equations in (40) are linearized near the up-streaming and the down-streaming. The determinant $\mathcal{D}$ takes the form

$$\mathcal{D}_0{}'=-2(B-1)(3B^2-6B+1) \quad (44)$$

near the up-streaming, and

$$\mathcal{D}_0{}'=\frac{1}{8}(3B-1)(29B^2-46B+5) \quad (45)$$

near the down-streaming. The acceptable solution are

$$B=0.1835 \text{ (or } M=1.8082) \quad (46)$$

for $\mathcal{D}_0{}'=0$, and

$$B=0.1174 \text{ (or } M=2.2609) \quad (47a)$$

$$B=0.3333 \text{ (or } M=1.3416). \quad (47b)$$

for $\mathcal{D}_0{}'=0$. Therefore, the lowest Mach number for the linearized equations to be unsolvable is 1.3416 which is the limit of Mach number in this case.

It is not difficult to show that original equations in (40) have solutions only when the Mach number is less than the limiting number. To do this, ones need to draw a singular surface given by $\mathcal{D}=0$ in the three dimensional phase space and examine the behavior of phase points of up-streaming and down-streaming flows. Figure 2 shows the surface $\mathcal{D}=0$ at $B=0.2$ plotted for $0<\tilde{P}<0.75$, $-0.4<\tilde{Q}<0.2$, and $-0.3<\tilde{R}^{(P)}<0$, by using the Plot3D of Mathematica. Similar shapes of surfaces are obtained for other values of B. In Figure 3, a contour curve (curve S) of the surface for $\tilde{R}^{(P)}=0$, and some hypothetical loci of solution points of (40) for $1<M<1.3416$ (curve a), $1.3416<M<1.8082$ (curve b), $1.8082<M<2.2609$ (curve c), $2.2609<M<\infty$ (curve d) are shown. It is evident that the original equations have solutions only when $1<M<1.3416$ in which the locus of solutions does not coincides with the singular surface.

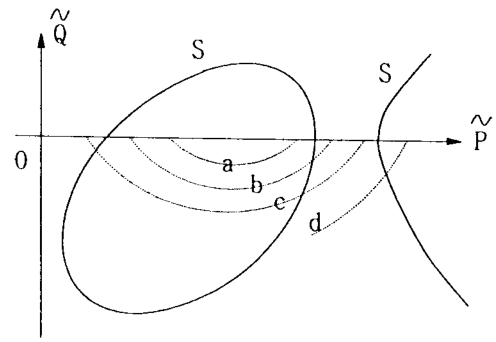

Figure 3. A shematic diagram of solutions of 20-moment equations in the phase space. The curve S represents a contour line of $\tilde{R}^{(P)}=0$ of the singular surface. The dotted curves are projections of hypothetical loci of solution points onto the $(\tilde{P}, \tilde{Q})$ plane in the phase space.

Discussions

In the present work, we have re-examined the limiting Mach number of Grad moment equations for stationary plane shock-waves. The method employed in the analysis is the linearization of equations near up-streaming and down-streaming flows. For the case of thirteen moment, it is confirmed that equations have solutions only when the Mach number is less than 1.6503 which has already been known since 1952. For twenty moment case, the limit of Mach number is decreased to 1.3416. This result implies that the limit of Mach number does not necessarily approach 1.851 as contrasted with the Holway's note.[8] As far as the convergence of series expansion is concerned, it is difficult to conclude that there exists an asymptotic limit of Mach number for the infinite series to converge.

It is worthwhile to notice that the thirteen moment equations do not reproduce the Navier-Stokes equation at the linear limit of up-streaming and down-streaming flows. The Navier-Stokes equation does not show the mathematical singularity for arbitrary Mach number. Although the equation do not predict experiments of strong shocks quantitatively, it describes the physical behavior of gases in near equilibrium. If a theory predicts correctly the whole shock profile from the up-streaming to the down-streaming, it should be reduced to the Navier-Stokes theory at near equilibrium regions. The discrepancy between the moment theory and the Navier-Stokes theory at near-equilibrium regions does not necessarily mean that the former is incorrect in the steep shock region also. It remains to be studied further that the thirteen moment equations can improve the prediction of Navier-Stokes theory in the steep shock zone for strong shock waves.[9]

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8. The number 1.851 in Holway's note (Reference 4) should be read 1.939. According to his idea, the number was calculated from the condition $T' \leq 2T'$ which means $\tilde{T}_0' \leq 2$ in present notation. By applying the condition to (9b), one obtains $M \leq 1.939$.
9. Ohr, Y. G. to be published in the *Bulletin of the Korean Chemical Society*.