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Graphical Estimation of the Parameters of the Stable Laws

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Abstract

This paper presents an easily used graphical procedure for simultaneous estimation of the index, skewness, scale, and location parameters of the stable laws. First, the index α and skewness β are estimated through the joint use of a tail length statistic $\widetilde{K}_t$ and a skewness statistic $\widetilde{K}_s$, both of which are functions of order statistics. Next, the function of order statistics needed for estimation of scale σ and location μ are determined from a nomogram indexed on the estimates of α and β. Some applications and examples are provided.

Key Words : Stable laws, order statistics, graphical estimation, stock price data, simulation.

1. Introduction

Estimation of the parameters of the stable laws has been discussed in detail by fama and roll(1971), DuMouchel(1971, 1975), Paulson, Holcomb, and Leitch(1975), Leitch and Paulson(1975), Wiener(1975), Koutrouvelis(1981), Fielitz and Rozelle(1981), McCulloch(1982), and Paulson and Delehanty (1985). Of the estimation procedures found in these articles, all but those given by Fama and Roll and McCulloch are computationally intensive. The

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Fama and Roll procedure is based on a few order statistics but is, however, concerned only with the estimation of the parameters of the symmetric stable laws.

The stable laws are most conveniently defined in terms of their characteristic function(Gnedenko and Kolmogorov, 1954, Ch. 5; Lukacs, 1970, Ch. 5).

$$\phi(u)=\exp\{i\mu u-|\sigma u|^{\alpha}(1+i\beta\frac{u}{|u|}x(u,\alpha))\} \tag{1.1}$$

where

$$x(u,\alpha)=\begin{cases}\tan\frac{\pi\alpha}{2} & \alpha\neq 1\\ \frac{2}{\pi}\log|u| & \alpha=1\end{cases}$$

u is a real number, $i^2=-1$, $0<\alpha\leq 2$, $|\beta|\leq 1$, $\sigma>0$, $|\mu|<\infty$. The parameter α, the index or characteristic exponent, governs the peackedness and the lenght of tail of the distribution determined by (1.1) : the lower the value of α relative to 2, the more peaked and the longer the tails of the distribution. The parameter β determines the skewness of the distribution : for $\beta<0$ and $\alpha\neq 1$, the distribution is skewed to the right; for $\beta>0$ and $\alpha\neq 1$, the distribution is skewed left; for $\beta=0$, the distribution is symmetric about the parameter μ. For $\beta=-1$ and $\alpha<1$, the distribution is skewed right and the random variable x defined by (1.1) is such that $x\geq\mu$ with probability unity, i.e., the distribution is one-sided; for $\beta=+1$ and $\alpha<1$, the distribution is skewed left with $x\leq\mu$ with probability unity. The parameter μ is not a true location parameter when (a) $\alpha=1$, $\beta\neq 0$, and (b) $\alpha<1$, $|\beta|=1$. When $\alpha>1$, μ is the mean of the distribution defined by (1.1). The parameter σ is the scale parameter of the distribution. The tails of the stable laws defined by (1.1) exhibit a Pareto behaviour, e.g., for $\alpha<2$, the right tail of the distribution function $S(x: \alpha, \beta, \sigma, \mu)$ of (1.1) satisfies

$$1-S(x: \alpha, \beta, \sigma, \mu) \sim c(\alpha, \beta,)\, x^{-\alpha}$$

for sufficiently large positive x where $c(\alpha, \beta)$ is a function depending only on α and β(Mandelbrot, 1963). A similar relationship holds for the left tail of S $(x: \alpha, \beta, \sigma, \mu)$.

The discussion of the above paragraph and the work of Fama and Roll

(1971) in the $\beta=0$ case suggests that, given a random sample $x_1, x_2, \cdots, x_n$ from $S(x: \alpha, \beta, \sigma, \mu)$, a few order statistics may contain much information concerning the parameters $\alpha, \beta, \sigma, \mu$. The extensive tables of the percentiles of $S(x: \alpha, \beta, \sigma, \mu)$, i.e., values x_p such that $S(x_p: \alpha, \beta, \sigma, \mu)=p$, $0\leq p\leq 1$, given by Paulson, Delehanty, and Brothers(1988) allowed us to investigate this suggenstion and we found that just a few order statistics are indeed very useful in the estimation of α, β, σ and μ. Furthermore, the graphical schemes which emerge from our work may sometimes also be employed to exclude the possibility that a set of data $x_1, x_2, \cdots, x_n$ can be effectively modeled by a stable law.

2. The Estimation Procedure for $(\alpha, \beta, \sigma, \mu)$

The tables of Paulson, Delehanty and Brothers(1988) provide the x_p of $S(x: \alpha, \beta, 1, 0)$ for $\alpha=.1(.1)1.9$, $\beta=-1(.1)0$, and values p of 0.0001(0.0001).001(.001).01(.01).99(.001).999(.0001).9999. These tables have allowed us to explore a variety of relationships in percentiles which might prove useful in the estimation of the parameters of the stable laws. This exploration indicated that much information concerning α and β is captured in the tail length and skewness functions

$$K_t=K_t(\alpha, \beta)=\log\frac{X_e-X_a}{X_d-X_b}, \tag{2.1}$$

$$K_s=K_s(\alpha, \beta)=\log\frac{X_e-X_c}{X_c-X_a}, \tag{2.2}$$

Figure 1 provides the graphical relationship among α, β, K_t, and K_s for a=0.05, b=0.3, c=0.5, d=0.7, e=0.95; for ease of disscussion we call this case A. Let $x_1, x_2, \cdots, x_n$ be random sample of size n from $S(x: \alpha, \beta, \sigma, \mu)$, Estimates for α and β, $\tilde{\alpha}$ and $\tilde{\beta}$ say, are obtained through (2.1) and (2.2) and Figure 1 as follows. Replace the value x_p in (2.1) and (2.2) by the sample estimate $\tilde{x}_p$, $0<p<1$. The sample estimate of x_p is given by $\tilde{x}_p=x_{(np)}$ where $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ are the sample values $x_1, x_1, \cdots, x_n$ arranged in ascending order. Here, $x_{(np)}$ is defined

for p=0.5

$$x_{(np)}=\begin{cases} x_{(n+1)/2} & \text{if n is odd} \\ (x_{(n/2)}+x_{(n/2+1)})/2 & \text{if n is even} \end{cases} \quad (2.3)$$

and for $p<0.5$

$$x_{(np)}=\begin{cases} x_{(np} & \text{if np is an integer} \\ (x_{(l)}+(np-l)(x_{(m)}-x_{(l)} & \text{if np is not an integer} \end{cases} \quad (2.4)$$

and where l and m are adjacent integers such that $l<np<m$. For $p>0.5$, we replace np by np+1 inthe right side expression of (2.4).

With x_p replaced by $\tilde{x}_p$, we obtain estimates $\tilde{K}_s$ and $\tilde{K}_t$ of K_s and K_t. References of the pair $(\tilde{K}_s, \tilde{K}_t)$ to Figure 1 yields a graphical estimates $(\tilde{\alpha}, \tilde{\beta})$ of the pair (α, β). A pair $(\tilde{K}_s, \tilde{K}_t)$ need not fall in the feasible region of Figure 1. The motivation behind K_t is easily obtained from Figure 1. The smaller the value of α relative to 2, longer the tail of $S(x: \alpha, \beta, \sigma, \mu)$ and hence the larger the value of K_t. As regards K_s, if $S(x: \alpha, \beta, \sigma, \mu)$ is skewed right, then $x_{.95}-x_{.5}$ will be larger than $x_{.5}-x_{.05}$ and K_s will be positive. The greater the positive skew, the larger the value of K_s. If $S(x: \alpha, \beta, \sigma, \mu)$ is symmetric, i.e., $\beta=0$, then $K_s=0$. Thus Figure 1 is applicable to the stable estimation problem only for $-1\leq\beta\leq0$. The modification needed for $|\beta|\leq1$ is given in section 3.

An examination of Figure 1 shows that the curves $\beta=-.9$ and -1 are nearly coincident for $.1\leq\alpha\leq1$ so that the percentiles $x_{.05}$ and $x_{.95}$ do not contain sufficient information to allow for discrimination in β when $-.9\leq\beta\leq-1$. Accordingly, we have also produced nomogram with a=0.01, b=0.3, c=0.5, d=0.7, e=0.99 in order to more use of the tail of the distribution. We call this the case B nomogram and is is given in Figure 2. Note that the $\beta=-.9$ and -1 curves are now well-seperated for $.1\leq\alpha\leq1$. The case B nomogram also contain greater separation between the α-cuves. Our experience indicates that both sets of nomograms or algorithms are useful in the following two ways. First, each algorithm provides a useful check on the other. Second, algorithm A should be used for moderate sample sizes, say $n\leq200$, while algorithm B should be used for $n\geq200$ although these recommended ranges are not meant to be hard and fast.

Some comments concerning the (α, β) nomograms are worth making explicit. First, the $\alpha=1$ curve is a limiting curve, defined by considerations of continuity which of course is not the case for the stable laws at $\alpha=1$ under the canonical form of the characteristic function (1.1) used here. The behavior of the stable laws defined by (1.1) in the neighborhood of $\alpha=1$ is governed by an essential singularity at $\alpha=1$, $\beta\neq0$. Thus discontinuous behavior in $S(x: \alpha, \beta, \sigma, \mu)$ is as follows. Let $\beta<0$ be fixed for $0<\alpha\leq2$. For $\alpha>1$ and $\alpha<1$ the distributions are skewed right. But for $\alpha=1$ and $\beta<0$, the distributions are skewed left. A similar behavor obtains for fixed $\beta>0$. For $\beta=0$, no discontinuity occurs as α passes through unity. Therefore our limiting curve was obtained by simply substituting the $\alpha=1$, $\beta>0$ curve in the polace of the $\alpha=1$, $\beta\leq0$ curve. Thus if $\widetilde{K}_s$ and $\widetilde{K}_t$ fall on the $\alpha=1$, $\beta\leq0$ curve (a probabilty zero event), the sign of β should be reversed. Estimation of σ and μ for $\alpha=1$ requires the special auxilialy nomograms given below. Second, as $\alpha\rightarrow2$ from below, all the β-curves converge to a single point. This convergence indicates the great difficulty in estimating both α and β for α near 2 and also the smooth convergence of the stable laws to the Gaussian law.

We now complete the graphical estimation procedure by providing first the μ and next σ nomograms. Given α and β, we find the value p for which x_p is zero, i.e., $\mu=0$, in the Paulson et al. tables of $S(x: \alpha, \beta, 1, 0)$. Given $x_1, x_2, \cdots, x_n$ from $S(x: \alpha, \beta, \sigma, \mu)$ with parameters α and β, then an estimator of μ is given by

$$\widetilde{\mu}=x_{(np)},\ \alpha\neq1 \tag{2.5}$$

Nomograms for determining p for estimation of μ are given in Figure 3.

Apart from the region of the parameter space for which $\alpha>0.8$ and $\beta<-0.8$, estimation for σ can be eveloped as follows. Given α and β, we find the values p and q for which $x_p=-1$ and $x_q=+1$ in the Paulson et al. tables of $S(x: \alpha, \beta, \sigma=1, \mu=0)$. The value p is that for which x_p is one σ unit below μ and the vlaue of q is that for which x_q is one σ unit above μ,. Nomograms for p and q are given in Figure 4. and 5. Given $x_1, x_2, \cdots, x_n$ form $S(x: \alpha, \beta, \sigma, \mu)$, determine estimates $\widetilde{\alpha}$ and $\widetilde{\beta}$ from nomogram A or B, then enter Figure 4 and Figure 5 to determine the percentiles p and q. An estimate of σ is given by

$$\widetilde{\sigma}=\frac{1}{2}(x_{(nq)}-x_{(np)}),\ \alpha\neq1, \tag{2.6}$$

and when we do not have simultaneously $\tilde{\alpha}<1$ and $\tilde{\beta}=-1$.
If $\tilde{\alpha}<1$ and $\tilde{\beta}=-1$ or $\tilde{\beta}$ is near -1, then an estimate for σ is given by

$$\tilde{\sigma}=x_{(nq)}-\tilde{\mu}, \qquad (2.7)$$

except for $\tilde{\alpha}>0.8$ and $\tilde{\beta}<-0.8$. In the latter case the values of q are very amall (but positive) and thus appear in Figure 5 to be 0 due to scaling considerations. For this case we have constructed the auxiliary nomogram given in Figure 6 which provides values of q for which $x_q=+K\ \sigma-$units above μ in the Paulson et al. tables of S(x: α, β, $\sigma=1$, $\mu=0$). Accordingly, estiamtes of σ in this case are given by

$$K\tilde{\sigma}=x_{(nq)}-\tilde{\mu},\ 0.8\leq\tilde{\alpha}<1,\ \tilde{\beta}<-0.8 \qquad (2.8)$$

where q is determined by value of K used in Figure 6 for the case K=6.

If $\alpha=1$, then estimates of σ and μ are obtained through the nomogram given in Figure 7. Given $\alpha=1$ and $\beta\leq0$, we find (a) the values p for which $x_p=0$ and (b) the values q and r for which $x_q=-1$ and $x_r=+1$ in the Paulson et al. tables. Then for values $\tilde{\alpha}=1$, $\tilde{\beta}<0$, estimates for σ and μ are determined form Figure 7 to be

$$\tilde{\sigma}=\frac{1}{2}(x_{(nr)}-x_{(nq)}), \qquad (2.9)$$

$$\tilde{\mu}=x_{(np)}. \qquad (210)$$

If $\tilde{\beta}$ is the estimate of β determined from Figure 1 or 2 when $\tilde{\alpha}=1$, then the proper estimate of β is $\tilde{\beta}=-\tilde{\beta}$ because of the discontinuous behavior of the stable distribution at $\alpha=1$, $\beta\neq0$.

Observe that the curves which eventuate the estimation of μ and σ given in Figure 3, Figure 4 and Figure 5 converge to a single point at (i) $\alpha=1$ and (ii) at $\alpha=2$. In case (i), this convergence of the curves to a single point at p=0, 1 and q=0, 1 represent the singularity in the behavior of the distributions at $\alpha=1$. At $\alpha=1$ these curves cannot be used to estimate μ or σ. For this case we have provided the auxiliary curves of Figure 7 through which the estimation may be effected. In case (ii), the p-curves in Figure 4 (b) converge to p=0.24, the q-curves in Figure 5(b) to q=0.76 and formular (2.6) is valid. The values p=0.24 and q=0.76 do not correspond to the .1587 and .8413 values usually associated with the Gaussian distribution because the usual statistical definition of Gaussian scale is different from the difinition of scale in the standard stable law literature. The convergence of the p and q curves in this case reflect the smooth convergence of the stable laws to the Gaussian law as $\alpha \to 2$.

Finally, regarding case (i), we remark that at $\beta=0$ p− and q−curves which eventuate estimation of σ are continuous across $\alpha=1$, and the p−curves of Figure 3 which determine the estimation relationship of μ are also continuous across $\alpha=-1$.

It is possible to develop more elaborate procedures for estimating α, β, σ and μ procedures which use a much larger number of order statistics. However, more elaborate schemes would obviate one of the objectives of the procedure embodied in Figures 1, 3, 4, 5, 6, 7 or 2, 3, 4, 5, 6, 7, namely that the procedure be easy to use and involve a minimum of computation.

3. Estimation for Positive β

The graphical estimation procedure of section 2 is based on the tables of Paulson et al. (1988) for $\beta \leq 0$ and as such is not applicable for $\beta > 0$. However, the procedure is easily extended to cover the entire range of β through the fundamental relationship (Zolotarev, 1964)

$$S(x: \alpha, \beta, \sigma, \mu) = 1 - S(-x: \alpha, -\beta, \sigma, -\mu) \tag{3.1}$$

This representation implies that to complete the estimation procedure for the entire range for the stable law parameters we should proceed as follows. If the sample value of $\widetilde{K}_s < 0$, transform the original sample $x_1, x_2, \cdots, x_n$ by $y_i = -x_i$, and use the nomograms as described in section 2. The estimators for the x−sample are then $\widetilde{\alpha}_x = \widetilde{\alpha}_y$, $\widetilde{\beta}_x = -\widetilde{\beta}_y$, $\widetilde{\sigma}_x = \widetilde{\sigma}_y$ and $\widetilde{\mu}_x = -\widetilde{\mu}_y$.

4. Application and Examples

The stable distributions have sometimes been used as models to describe rates of return on common stocks. In illustration of the graphical procedure we estimate the parameters of the stable laws for the log price relatives of twenty five stocks listed in Table 1 and selected at random from New York Stock Exchange listings for period 1962−1969. The first entry provides the parameter estimates for the highly efficient Paulson and Delehanty (1985) procedure. The agreement is very good and all fall in the feasible region.

The literature on common stock price behavior (Fama and Roll, 1971) also suggests that the behavior of log price relatives may be modeled as a mixture of normals. Our interest here is to determine whether samples of mixtures, e.g., 50% N(0, 1) and 50% N(0, 9) would return stable law parameter estimates on application of the graphical procedure. One hundred samples of size 1000 and samples of size 10,000 were generated from 50% N (0, 1) and 50% N(0, 9). In each case the values $\widetilde{K}_s$ and $\widetilde{K}_t$ fell within the stable feasible region. This finding indicates that the graphical estimation procedure cannot generally be used as a test of fit for the composite hypothesis of stable distribution and that more sensitive tools such as the graphical method based on domains of attraction of Paulson and Royyuru(1986) must be used.

However, when the graphical procedure is applied to 100 samples of size 200 and 100 samples of size 1,000 from each of the lognormal (0, 1), gamma (α, 1), α=0.5, 1, 2, 4, the sample K_s and K_t values all fell outside the feasible region. Accordingly, the graphical procedure of section 2 has uses that extend to model rejection as well as estimation.

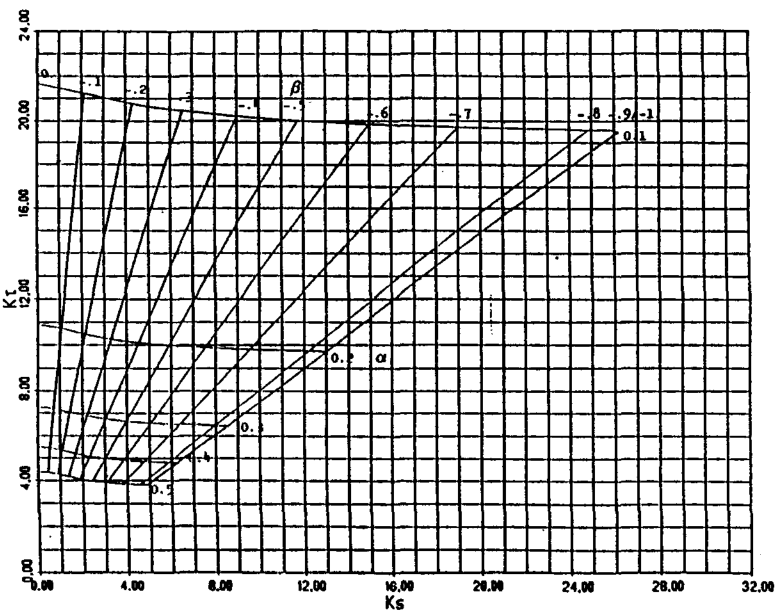
In order to illustrate the use of the graphical estimation procedure, we simulate some random samples of size n=100, 200, 400 and 1000 from stable distributions with fixed parameter values by the algorithm of Chambers, Mallows, and Stuck(1976). The case B figures are used to produce estimates of α, β, σ, μ. Our detailed explanation will be restricted to the sample size n=1000. Table 2 contains all results and indicates in particular the typical variability and might experience in estimation of parameters from stable distributed samples.

Exapmle 1. Here we simulate variates from a stable distribution with parameter vector(α, β, σ, μ)=(1.5, 0.5, 1.0, 0). From the ordered sample of x−values we obtain the values $\widetilde{K}_t$=2.08 and $\widetilde{K}_s$=−0.626 respectively. Since $\widetilde{K}_s$ has a negative value, by taking the negative of the variates as in section 3, we obtain $\widetilde{K}_s$=0.626. Then from Figure 2 we obtain the estimates $\widetilde{\alpha}$=1.61 and $\widetilde{\beta}$=−0.52. From Figure 3 we obtain the percentile value p=0.57 which results in an estimate $\tilde{\mu}=0.672$ from (2.4) for $p>0.5$ case and (2.5). From Figure 4 and Figure 5 we obtain percentile values $p=0.29$ and $q=0.79$ which results in $x_{(np)}=-0.416$ and $x_{(nq)}=1.50$. From (2.6) we find $\tilde{\sigma}=0.96$. By the results of section 3 we have finally $\tilde{\alpha}_x=1.61$, $\tilde{\beta}_x=+0.52$, $\tilde{\sigma}_x=0.96$, $\tilde{\mu}_x=-0.672$.

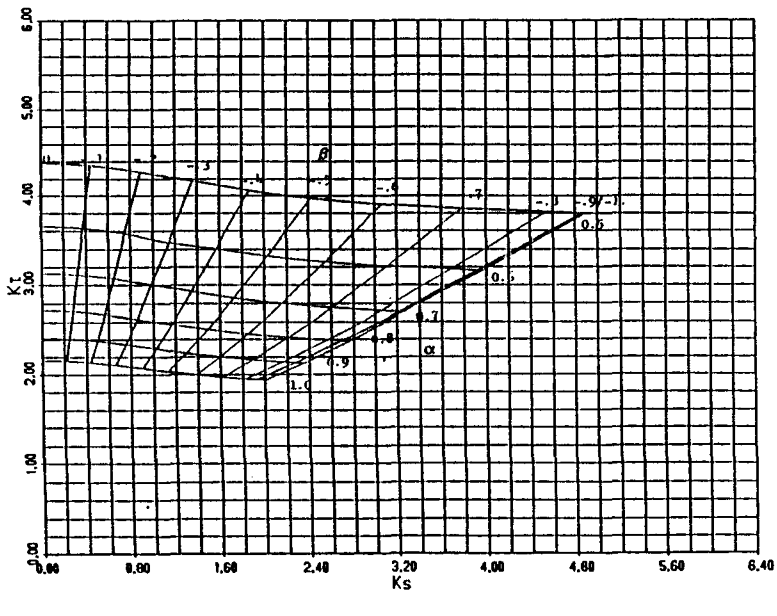
Example 2. Here we simulate variates from a stable distribution with parameter vector $(\alpha, \beta, \sigma, \mu)=(0.5, -1, 1.0, 0)$. We find $\tilde{K}_t=7.65$ and $\tilde{K}_s=8.79$, from Figure 2 we find the estimates $\tilde{\alpha}=0.47$ and $\tilde{\beta}=-1$. From Figure 3 we get $p=0$. Thus the estimate of μ for this sample is the minimal order statistic, i.e. $\tilde{\mu}=0.086$. From Figure 5 we get $q=0.337$ and thus compute $x_{(nq)}=1.087$. From (2.6) we obtain $\tilde{\sigma}=1.087-0.086=1.001$.

Example 3. For this sample we simulate variates from a stable distribution with parameter vector $(\alpha, \beta, \sigma, \mu)=(0.9, -0.9, 1.0, 0)$. We compute $\tilde{K}_t=4.03$ and $\tilde{K}_s=3.72$, and from Figure 2 we obtain $\tilde{\alpha}=0.87$ and $\tilde{\beta}=-0.94$. With these values of $\tilde{\alpha}$ and $\tilde{\beta}$ we find $p=0.005$ which leads to an estimate of μ as $\tilde{x}_{(np)}=\tilde{\mu}=-2.3$. From Figure 6 we find $p=0.543$ and from (2.8) we obtain $6\tilde{\sigma}=8.807$ or $\tilde{\sigma}=1.47$.

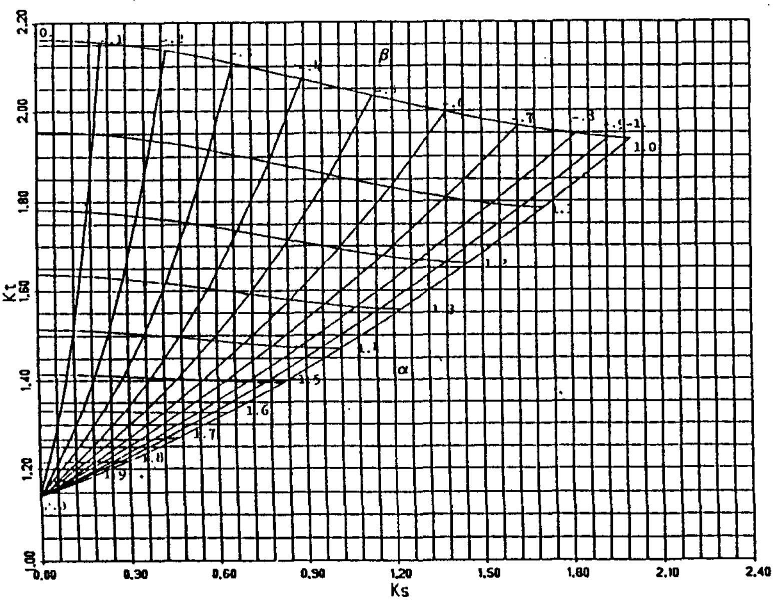
Example 4. In this final case we simulate variates from a stable distribution with $(\alpha, \beta, \sigma, \mu)=(1, -0.5, 1.0, 0)$. We find that $\tilde{K}_t=3.58$ and $\tilde{K}_s=-1.34$. For this pair $\tilde{\alpha}$ is very near unity with $\tilde{\beta}=-0.57$. Figure 7 provides a value $p=0.57$ which leads to the estimate $\tilde{\mu}=0.07$. We find $r=0.845$, $q=0.345$, $\tilde{x}_{(nr)}=1.071$, $\tilde{x}_{(nq)}=-0.880$. From (2.9) we obtain $\tilde{\sigma}=0.98$.



(a) α=0.1−0.5

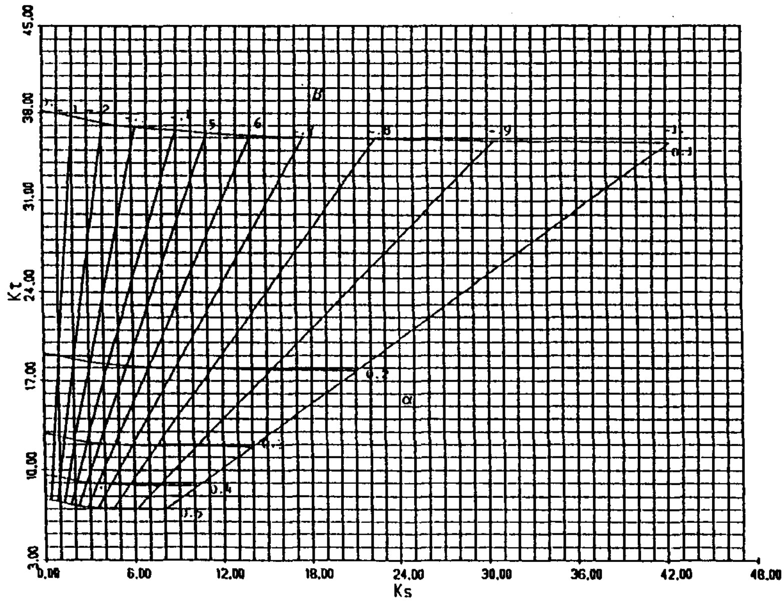


(b) α=0.5−1.0

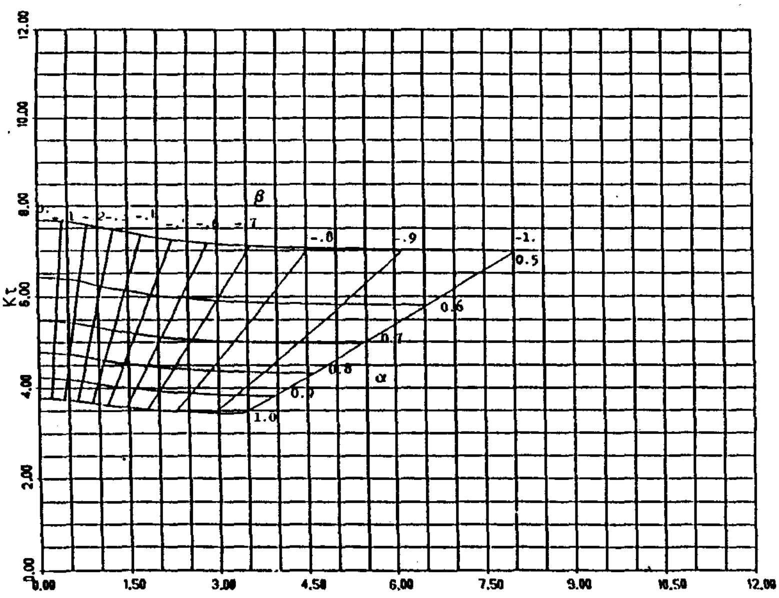


(c) α=1.0−2.0

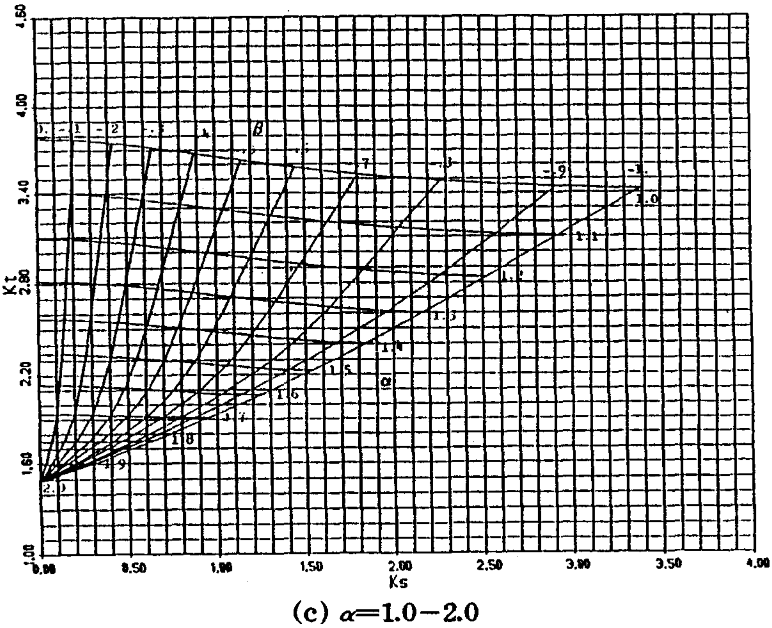
Figure 1. $\widetilde{\alpha}$, $\widetilde{\beta}$ **by** K_s **and** K_t **(Case A)**



(a) α=0.1−0.5

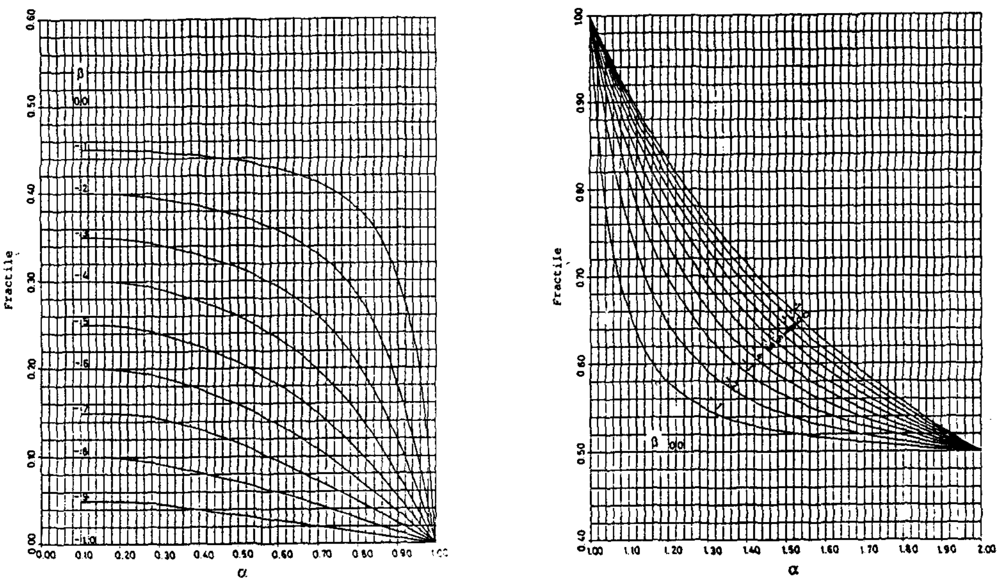


(b) α=0.5−1.0



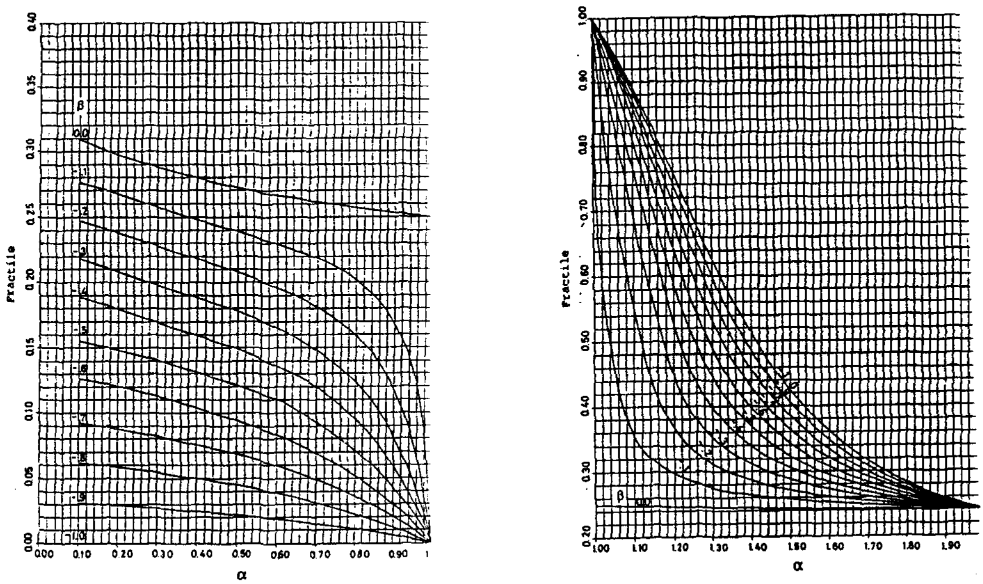
(c) α=1.0−2.0

Figure 2. $\widetilde{\alpha}$, $\widetilde{\beta}$ by K_s and K_t(Case B)



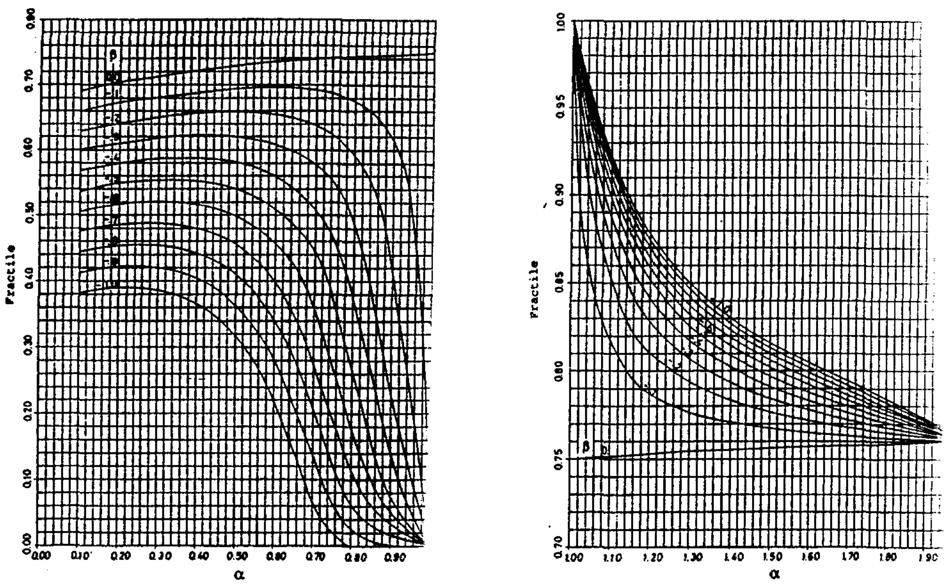
(a) α=0.1−1.0 (b) α=1.0−2.0

Figure 3. Fractile p of μ, give α, β



(a) α=0.1−1.0 (b) α=1.0−2.0

Figure 4. Fractile p of σ for the case x_p=−1



(a) α=0.1−1.0 (b) α=1.0−2.0

Figure 5. Fractile q of σ for the case x_q=+1

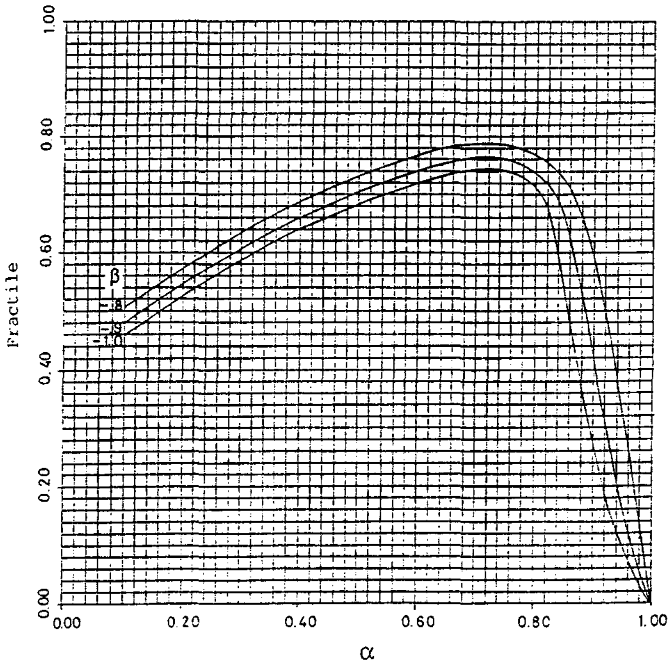


Figure 6. Fractile q of σ for the case x_q=+6

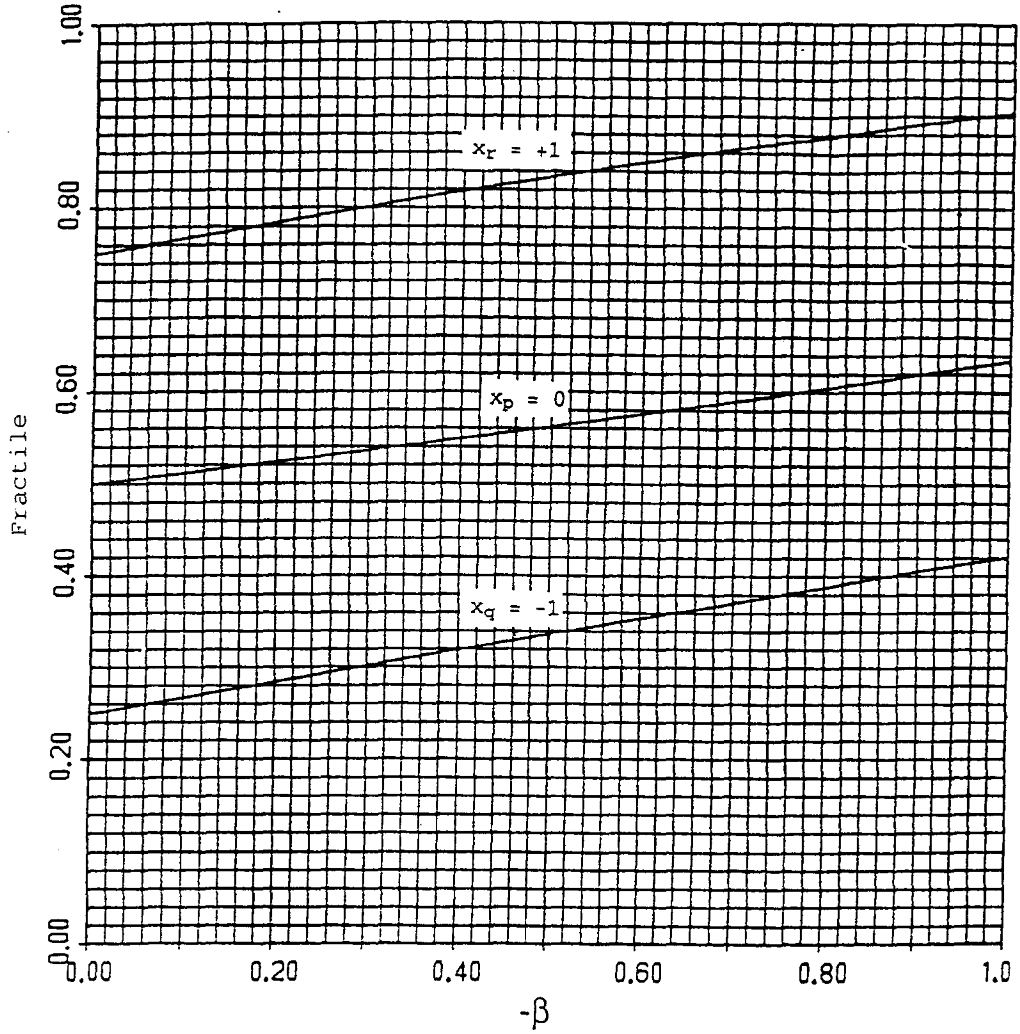


Figure 7. Fractiles p, q and r of σ for the cases $x_p=0$, $x_q=-1$, $x_r=+1$ when $\alpha=1$, $\beta<0$

Table 1. Parameter estimates of 25 stocks

Security	α	β	σ	μ
1. American bakeries	1.62	−0.11	1.04(−2)	8.69(−5)
	1.63	−0.03	1.05(−2)	0.
2. AMP, Inc.	1.77	−0.23	1.15(−2)	1.16(−3)
	1.77	−0.08	1.05(−2)	0.
3. Armstrong Cork	1.81	−0.22	8.52(−3)	6.68(−4)
	1.75	−0.16	7.94(−3)	0.
4. U.S. Gypsum	1.73	−0.05	8.06(−3)	−3.01(−5)
	1.77	−0.08	7.46(−3)	0.
5. Western Pacific R.R.	1.72	−0.22	8.09(−3)	3.30(−4)
	1.77	−0.13	7.03(−3)	0.
6. Bendix	1.67	−0.17	9.05(−3)	5.42(−4)
	1.70	−0.06	8.58(−3)	0.
7. Bobbie Brooks	1.65	−0.22	1.12(−2)	3.76(−4)
	1.63	−0.06	1.06(−2)	0.
8. J.I. Case	1.72	−0.48	1.58(−2)	1.40(−3)
	1.78	−0.26	1.46(−2)	0.
9. Colgate-Palmolive	1.83	−0.20	8.79(−3)	5.97(−4)
	1.75	−0.16	8.66(−3)	0.
10. Cone Mills	1.70	−0.31	9.76(−3)	4.48(−4)
	1.72	−0.26	1.02(−2)	0.
11. Curtiss-Wright	1.62	−0.63	1.16(−2)	1.38(−3)
	1.72	−0.43	1.23(−2)	0.
12. Delta Air Lines	1.77	−0.25	1.28(−2)	1.77(−3)
	1.75	−0.09	1.24(−2)	0.
13. Max Factor	1.73	−0.21	9.73(−3)	8.40(−4)
	1.70	−0.15	8.93(−3)	0.
14. General Amer. Investors	1.77	0.03	6.91(−3)	5.43(−4)
	1.70	−0.10	6.30(−3)	0.

(Table 1. cont.)

15. General Dynamics	1.77	−0.14	8.43(−3)	1.09(−4)
	1.76	−0.25	1.17(−2)	0.
16. Green Shoe Mfg.	1.71	−0.14	8.43(−3)	1.09(−4)
	1.64	−0.12	7.99(−3)	0.
17. Int'l. Harvester	1.82	−0.17	7.61(−3)	1.70(−4)
	1.74	−0.11	6.93(−3)	0.
18. Lab for Electronics	1.76	−0.44	1.94(−2)	9.75(−4)
	1.76	−0.17	1.83(−2)	0.
19. Libbey-Owens-Ford	1.69	−0.04	5.91(−3)	−9.63(−6)
	1.75	−0.03	5.19(−3)	0.
20. McQuay-Norris	1.61	−0.06	8.45(−3)	−2.74(−5)
	1.64	−0.18	6.63(−3)	0.
21. National Fuel Gas	1.85	−0.06	7.01(−3)	9.38(−5)
	1.74	−0.04	7.50(−3)	0
22. Procter & Gamble	1.74	−0.21	5.64(−3)	2.67(−4)
	1.75	−0.06	5.52(−3)	0.
23. Ryder System	1.81	−0.83	1.52(−2)	1.83(−3)
	1.77	−0.38	1.39(−2)	0.
24. Sperry System	1.70	−0.15	1.21(−2)	1.23(−3)
	1.73	−0.12	1.37(−2)	0.
25. Texas Instruments	1.70	−0.15	1.21(−2)	1.23(−3)
	1.71	0.01	1.13(−2)	0.

Table 2. Estimates by graphical procedure

parameters\size	100	200	400	1000
α=1.5	1.64	1.70	1.76	1.61
β=0.5	0.86	0.04	0.78	0.52
σ=1	0.94	1.02	0.98	0.96
μ=0	−0.63	−0.49	−0.74	−0.67
α=1.5		0.39	0.49	0.47
β=−1.0	(out of range)	−0.93	−0.75	−1
σ=1		1.41	1.43	1.
μ=0		0.19	0.32	0.09
α=0.9	1.14	0.76	0.82	0.87
β=−0.9	−0.985	−0.95	−0.97	−0.94
σ=1	1.39	0.21	0.92	1.47
μ=0	10.07	4.01	2.71	−2.3
α=1	1.33	1.17	1.18	1.
β=−0.5	−0.77	−0.08	−0.51	−0.56
σ=1	0.55	1.07	0.36	0.98
μ=0	0.42	0.22	0.84	0.07

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