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A GLOBAL STUDY ON SUBMANIFOLDS OF CODIMENSION 2 IN A SPHERE

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ABSTRACT. M be an $n(\geq 3)$-dimensional compact connected and oriented Riemannian manifold isometrically immersed on an $(n+2)$-dimensional sphere $S^{n+2}(c)$. If all sectional curvatures of M are not less than a positive constant c, show that M is a real homology sphere.

0. Introduction

Let M be an $n-$dimensional compact connected and oriented Riemannian manifold isomerically immersed in an $(n+2)-$dimensional Euclidean space R^{n+2}. As is well known, if M is of positive curvature, then M is a homotopy sphere [4]. This result is generalized by Baldin and Mercuri [2], Baik and Shin [1] in the case of non-negative curvature, which is stated as follows : if M is of non-negative curvature, then M is either a homotopy sphere or diffeomorphic to a product of two spheres. In particular, if there is a point at which of positive curvature, then M is a homeomorphic to a sphere. This is a kind of reports which is devoted to study on a submanifolds of codimension 2 in a sphere $S^{n+2}(c)$. In the last section we prove the following :

Theorem 0.1. *Let M be an $n(\geq 3)-$dimensional compact connected and oriented Riemannian manifold isometrically immersed on an $(n+2)-$dimensional sphere $S^{n+2}(c)$. If all sectional curvatures of M are not less than a positive constant c, then M is a real homology sphere.*

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1. Associative curvature forms

Let V and W be real vector spaces of finite dimension n and p respectively, and B be a symmetric bilinear map of $V \times V$ into W. Suppose that $n > 2$ and W has an inner product $<,>$. Define the associative curvature form $R_B : \wedge^2 V \times \wedge^2 V \rightarrow R$ by

$$R_B(x \wedge y, z \wedge w) =< B(x,z), B(y,w) > - < B(x,w), B(y,z) > . \tag{1.1}$$

for any vectors x, y, z and w in V. The map R_B is again symmetric and hence the eigenvalues of R_B is all real. R_B is said to be positive definite or positive semi-definite according as all eigenvalues of R_B are positive or non-negative, respectively. Next, we define the associative sectional curvature form K_B by

$$K_B(x,y) = R_B(x \wedge y, x \wedge y) \tag{1.2}$$

whenever $x \wedge y \neq 0$. The map K_B is said to be positive definite or positive semi-definite according as $K_B(x,y)$ is positive or non-negative for linearly independent vectors x and y in V, respectively.

Consider the following conditions for the bilinear map B :

(1) There exists an orthonormal basis $\{\xi_{n+1}, \cdots, \xi_{n+p}\}$ of W in such a way that the real valued function $H_a(x,y)$ on $V \times V$ defined by $H_a(x,y) =< B(x,y), \xi_a >$ is non-negative for any indices $a = n+1, \cdots, n+p$.

(2) R_B is positive semi-definite.

(3) K_B is positive semi-definite.

Lemma 1.1. *(1) → (2) → (3) In particular, if* $p = 2$, *the conditions are all equivalent.*

Proof. we prove the assertion (1)→(2). Suppose the condition (1) holds. By making use of the function H_B for an orthonormal basis $\{\xi_a\}$ an image of B is given by $B(x,y) = \sum_a H_a(x,y)\xi_a$. Then we get

$$R_B(x \wedge y, z \wedge w) = \sum_a (H_a(x,z)H_a(y,w) - H_a(x,w)H_a(y,z)) \tag{1.3}$$

and we have $R = \sum_a R_a$. In order to prove that R_B is positive semi-definite, it suffices to show that all the map R_a are positive semi-definite. For a fixed index

a, let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for V which diagonalizes the function H_a, namely, $H_a(e_i, e_j) = \lambda_i \delta_{ij}$. Here and in the sequel, indices i and j run over the range $\{1, \cdots, n\}$ and an index a run over the range $\{n+1, \cdots, n+p\}$, unless otherwise stated $\lambda_i > 0$ for all indices i, because H_a is positive semi-definite. Since the inner product $<, >$ of $\wedge^2 V$ is by definition

$$< x \wedge y, z \wedge w > = < x, z >< y, w > - < x, w >< y, z >,$$

then the function (1,3) of the function R_a implies

$$R_a(e_i \wedge e_j, e_k \wedge e_l) = \lambda_i \lambda_j < e_i \wedge e_j, e_k \wedge e_l > .$$

It means that $\{e_i \wedge e_j : i < j\}$ forms an orthonormal basis for $\wedge^2 V$ which diagonalizes R_a with eigenvalues $\lambda_i \lambda_j (\geq 0)$. So R_a is positive semi-definite.

Next the assertion (2) $\rightarrow$ (3) is trivial. In the case where $p = 2$, it only remains to prove that the condition (3) implies the condition (1).

Suppose that the map K_B is positive semi-definite. Then for all pair (x, y) of linearly independent vectors, we have

$$K_B(x, y) = < B(x, x), B(y, y) - \|B(x, y)\|^2 > 0, \tag{1.4}$$

where $\| \quad \|$ means the norms for the vector space W. Now there might exist a non-asymptotic vector x in $V - \{0\}$. Suppose that any vector x in $V - \{0\}$ is asymptotic. Then $H_a(x, x)$ must be equal to zero, because of $H_a(x, x) = < B(x, x), \xi_a >$ for any orthonormal basis $\{\xi_a\}$ for W. If this case can be regarded as the special one of positive semi-definiteness, then it is nothing but the condition (1). Choose an orientation for W, and for fixed vector x_0 and any vector x in $V - \{0\}$, let $\theta(x)$ denote an angle from $B(x_0, x_0)$ to $B(x, x)$. $\theta(x)$ is defined only module 2π but it follows from (1.4) that θ is continuous function of $V - \{0\}$ into the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For a unit sphere S of V centered with origin the restriction of θ to S is also continuous, so it must attain its maximum θ_1 and minimum θ_2. Again, taking the inequality (1,4) into account, we get $\theta_1 - \theta_2 \leq \frac{\pi}{2}$.

Let $\bar{\theta} = (\theta_1 + \theta_2)/2, \bar{\theta_1} = \bar{\theta} + \frac{\pi}{4}$ and $\bar{\theta_2} = \theta - \frac{\pi}{4}$, and $\xi(\theta)$ be a unit vector in W to which the direct angle from $B(x_0, x_0)$ is equal to θ. Then by putting $\xi_{n+1} = \xi(\theta_1)$ and $\xi_{n+2} = \xi(\theta_2)$, $\{\xi_{n+1}, \xi_{n+2}\}$ is an orthonormal basis for W, and by choosing the angle θ_1 and θ_2 it turns out that

$$\bar{\theta_2} \leq \theta_2 \leq \theta(x) \leq \theta_1 \leq \bar{\theta_1}$$

for any vector x in S. This implies that the angle between $\xi(\theta(x))$ and $\xi_\alpha(\alpha = n+1, n+2)$ is less than or equal to $\frac{\pi}{2}$ for any x in S, and so is the angle between $B(x,x)$ and ξ_α, because of $B(x,x) = \|B(x,x)\|\xi(\theta(x))$.

Thus the forms H_a are both positive semi-definite. This concludes the proof.

2. Curvature operator

In this section, the concept of the curvature operator in a Riemannian manifold (M, g) will be introduced and the manifold structures of M which are influenced by some conditions of the operator are investigated.

For a point x in M, let R_x be an associated curvature operator. A linear map P^* of $\wedge^2 M_x$ into $\wedge^2 M_x^*$ for any point x in M is defined by $u \wedge v \to R(\cdots, u, v)$ and by this duality an endomorphism p_x of $\wedge^2 M_x^*$ into itself is manifactured. It turns out that p_x satisfies

$$< p_x(u^* \wedge v^*), w^* \wedge z^* >=< p_x^*(u,v), w^* \wedge z^* >= R_x(u,v,w,z) \tag{2.1}$$

for any vectors u, v, w and z in M_x, where μ^* denotes the dual form in M_x associated with the vector u. The operator p_x is called a curvature operator at x. Since p_x is the symmetric operator, each eigenvalue of it is real. If all eigenvalues of p_x are contained in the interval $[\lambda, \mu]$, then one says $\lambda \leq p_x \leq \mu$, and if for any point x on M this property is satisfied, then $p(M)$ is a set which consists of all curvature operators at all points in M.

Now, for an orthonormal basis $\{u_1, \cdots, u_n\}$ of M_x and its dual basis $\{w^1, \cdots, w^n\}$ for M_x^* relative to $\{u_1, \cdots, u_n\}$, the following equation is given :

$$< p_x(w^i \wedge w^j), w^i \wedge w^j >= R(u_i, u_j, u_i, u_j) = g(R(u_i, u_j)u_i, u_j), \tag{2.2}$$

from which

$$< p_x(w^i \wedge w^j), w^i \wedge w^j >= k(u_i, u_j), \tag{2.3}$$

where $k(u_i, u_j)$ means a sectional curvature of a plane section spaned by the orthonormal vectors u_i and u_j. It follows that $K(M) \geq 0$ if $p(M) \geq 0$. Under the pinching of the curvature operator $p(M)$, the curvature tensor R and the Ricci tensor S are also pinched as follows :

$$\begin{aligned} \lambda(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) \leq g(R(u_i,u_j)u_k,u_l) \leq \mu(\delta_{jl}\delta_{ik} - \delta_{ik}\delta_{jl})\lambda(n-1)\delta_{ij} \\ \leq S(u_i,u_j) \leq \mu(n-1)\delta_{ij}. \end{aligned} \tag{2.4}$$

Thus, if $\lambda \le p(M) \le \mu$. Remark here that the converse is not necessarily true.

Now, it plays an important role to restrict with the manifold structures of M that the curvature operator $p(M)$ is pinched. This is first studied by Yano and Bochner [5]. Suppose that $\lambda \le p(M) \le \mu$. Given any p-form w in $\wedge^p M_x^*$, we put

$$\begin{aligned} F(w) = & \sum_{i,j} \sum_{i_2 \cdots i_p} S(i,j) w(j, i_2, \cdots, i_p) \\ & - \frac{p-1}{2} \sum_{i,j,k,l} \sum_{j, i_3, \cdots, i_p} R(i,j,k,l) w(i,j,i_3, \cdots i_p) \cdot w(k,l,i_3,\cdots,i_p) \end{aligned} \tag{2.5}$$

then the fuction $F(w)$ can be bounded from below. Namely, it follows from (2,4) that $F(w) \ge \{(n-1)\lambda - (p-1)\mu\}\{w\}^2$.

This implies $F(w) > 0$ if $\lambda = \frac{\mu}{2}$ and $2p < n+1$.

In order to generalize the theorem due to Yano and Bochner, the other expression of the function F will be considered by making use of the curvature operator since components of any p-form w in $\wedge^p M_x^*$ with respect to the orthonormal basis $\{u_1, \cdots, u_n\}$ for M_x are given by $w(i_1, \cdots, i_p)$, where $\{w^{i_1} \wedge \cdots \wedge w^{i_p}\}(i_1, \cdots, i_p \in \{1, 2, \cdots, n\})$ is an orthonormal basis of $\wedge^p M_x^*$, the $p-$form w is expressed by

$$w = \sum_{i_1, \cdots i_p} w(i_1, \cdots\cdots, i_p) w^{i_1} \wedge \cdots\cdots \wedge w^{i_p}$$

For a $p-$form w at x we shall consider a family of exterior $2-$forms $(i_1, \cdots, i_p)^w$ corresponded to the $p-$form w, which is defined by

$$(i_1, \cdots, i_p)^w = \sum_{k=i}^{p} \sum_{j_k=l}^{n} w(i_1, \cdots, i_{k-1}, j_k, i_{k+1}, \cdots, i_p) w^{j_k} \wedge w^{i_k}. \tag{2.6}$$

Moreover a family of scalars $(i_1, \cdots, i_p)^{\theta(w)}$ associated with the form w is produced.

The scalar is also defined by

$$(i_1, \cdots, i_p)^{\theta(w)} = < p_x (i_1, \cdots, i_p)^w, (i_1, \cdots, i_p)^w > . \tag{2.7}$$

We have by (1,4)

$$F(w) = A(w) - \frac{p-1}{2} B(w), \tag{2.8}$$

where $A(w) = \sum_{i,j} \sum_{i_2, \cdots, i_p} S(i,j)\, w(j, i_2, \cdots, i_p)\, w(j, i_2, \cdots, i_p)$,

$B(w) = \sum_{i,j,k,l} \sum_{i_3, \cdots, i_p} R(i,j,k,l)\, w(i,j,i_3,\cdots,i_p)\, w(k,l,i_3,\cdots,i_p)$.

The following Lemma 2.1 and Lemma 2.2 are due to Meyer [3].

Lemma 2.1. $F(w) = \frac{1}{p}\sum_{i_1,\cdots,i_p}(i_1,\cdots,i_p)^{\theta(w)}$

Lemma 2.2. *If w is an exterior $p-$form on M which does not vanishes at x for $i \leq p \leq n-1$, then the associated $2-$form is not equal to zero at x.*

By making use of Lemmas 2.1 and 2.2, the following property is verified.

Theorem 2.3. *Let M be an $n-$dimensional compact and oriented Riemannian manifold. If all curvature operators satisfy $p(M) > 0$, then M is a real homology sphere*

Proof. The hypothesis $p(M) > 0$ implies that for a point x all eigenvalues of the operator p_x are positive, by (2,7) any exterior $p-$form w satisfies the condition

$$(i_1,\cdots,i_p)^{\theta(w)} \geq 0$$

for any indices $i_1,\cdots,i_p$. It follows from Lemma 2.1 that $F(w) \geq 0$.

It implies that in the equation

$$(\triangle w, w) = \|\nabla w\|^2 + Q(w),$$

where $Q(w) = \int_M F(w)dV_M, \triangle w$ is the Laplacian of w and ∇w is the covariant derivative of w, the second term $Q(w)$ of the right hand side is positive. If the $p-$form w is harmonic, then triangle $w = 0$ and we obtain that $F(w)$ and $\triangle w$ vanish everywhere on M. Thus the $p-$form is parallel. Since w is parallel, the norm $\|w\|$ vanishes everywhere on M. Therefore, by the Theorem due to Hodge the $p-$th homology group H^p satisfies

$$H^p(M,R) = 0, 0 < p < n$$

This completes the proof.

3. Proof of Theorem 0.1.

Let $S^{n+2}(c)$ be an $(n+2)-$dimensional sphere of constant curvature. Let i be an isometrically immersion of an $n-$dimensional compact and oriented Riemannian manifold M into the sphere $S^{n+2}(c)$. For any point of M we shall denote $i(M)$ on $S^{n+2}(c)$ by the same symbol x, since there is no danger of confusion and moreover the computation is local. Futhermore, a tangent vector u at x is identified with $d_{i_x}(u)$. Then the tangent space M_x at x is a subspace of the tangent space $\bar{M}_x$ of ambient space $\bar{M} = S^{n+2}(c)$ at x.

Let N_x be the orthogonal complement of M_x in $\bar{M}_x$, which is called a normal space to M at x. Let H be the second fundamental form of the immersion i. For the triple (M_x, N_x, H_x) at each point x in M, (algebraic preliminaries which prepared) for section 1 can be applied.

Let R_B be the associated curvature form on M_x which is defined by (1,1) and K_B be the real valued map on $M_x \times M_x$ defined by (1,2). From (1,1) we have

$$R_B(u \wedge v, w \wedge z) = R(u, v, w, z) - (< u, w >< v, z > - < u, z >< v, w >), \quad (3.1)$$

where R denotes the Riemannian curvature tensor on M. Then we get

$$K_B(u, v) = (K(u, v) - c)(\|u\|^2 \|v\|^2 - < u, v >^2), \quad (3.2)$$

where $K(u, v)$ is the sectional curvature of plane spaned by linearly independent vectors u and v on M_x. By the assumption of the Theorem 0.1, it follows that $K_B \geq 0$ from (3,2). Thus, by the Lemma 1.1 the associated curvature from $R_B \geq 0$. Hence the curvature form p_x at x of M satisfies $p_x \geq c$, because of (2,1). Then we have $p(M) \geq c \geq 0$.

By the Theorem 2.1, M is a real homology sphere. This completes the proof.

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