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ON THE SOME PROPERTIES OF THE LIMITS SETS

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ABSTRACT. In this paper, we investigate the properties of various limit sets. In particular, we study the relationship between the recurrent set and special γ-limit set. And also we show that if x is not almost periodic, then x is special α-limit.

1. Introduction

Let I be the unit interval, S^1 the circle and X be a compact metric space. And let $C^0(X, X)$ denote the set of continuous maps from X into itself.

Let $f \in C^0(X, X)$. For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^{n+1} = f \circ f^n$. Let f^0 denote the identity map of X. The *forward orbit* $Orb(x)$ of $x \in X$ is the set $\{f^k(x) \mid k = 0, 1, 2, \cdots\}$. Usually the forward orbit of x is simply called the *orbit* of x.

For any continuous map f from a compact metric space X to itself, throughout this paper, P, AP, R and $S\Gamma$ denote the set of periodic points, almost periodic points, recurrent points and specialγ-limit points of f, respectively. And for any set Y, $\overline{Y}$ denotes the closure of Y as usual.

In this paper, we study the relationship between the recurrent set and special γ-limit set. And also we show that if x is not almost periodic, then x is special α-limit. In fact, we obtain the following results :

Theorem A. *Let f be a continuous map of the circle S^1 to itself. Suppose that P is empty. Then $\overline{R} \subset S\Gamma$.*

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Theorem B. *If* $x \in \overline{P}_- \cap \overline{P}_+$, *then* $x \in s\alpha(x)$. *If in addition* x *is not almost periodic, then for some* $\epsilon > 0$, *either* $x \in s\alpha(y)$ *for every* $y \in (x-\epsilon, x]$ *or* $x \in s\alpha(y)$ *for every* $y \in [x, x+\epsilon)$.

2. Basic concepts

Let (X,d) be a compact metric space and $f \in C^0(X,X)$. A point $x \in X$ is called a *periodic point* of f if for some positive integer n, $f^n(x) = x$. The period of x is the least such integer n. We denote the set of periodic points of f by P.

A point $x \in X$ is called a *recurrent point* of f if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent points of f by R.

A point $y \in X$ is called an *ω-limit point* of x if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of ω-limit points of x by $\omega(x)$.

A point $y \in X$ is called an *α-limit point* of x if there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{y_i\}$ of points such that $f^{n_i}(y_i) = x$ and $y_i \to y$. The symbol $\alpha(x)$ denotes the set of α-limit points of x.

A point $y \in X$ is called a special *α-limit point* of x if there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{y_i\}$ of points such that

(1) $x = y_0$,

(2) $f^{n_i}(y_i) = y_{i-1}$,

(3) $\lim_{i\to\infty} y_i = y$.

The symbol $s\alpha(x)$ denotes the set of special α-limit points of x and $SA = \bigcup_{x \in X} s\alpha(x)$.

A point $y \in X$ is called a special *γ-limit point* of x if $y \in \omega(x) \cap s\alpha(x)$. The symbol $s\gamma(x)$ denotes the set of special γ-limit points of x and $S\Gamma = \bigcup_{x \in X} s\gamma(x)$.

3. Main Results

The following lemma founded in [BCMY].

Lemma 1. *Let $f \in C^0(S^1, S^1)$ and $I = [a, b]$ be an arc for some $a, b \in S^1$ with $a \neq b$, and let $I \cap P = \phi$.*

(a) Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x < f(x)$. Then

(i) if $y \in I, x < y$ and $f(y) \notin [y, b]$, then $[x, y]$ f-covers $[f(x), b]$, and

(ii) if $y \in I, y < x$ and $f(y) \notin [y, b]$, then $[y, x]$ f-covers $[f(x), b]$.

(b) Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x > f(x)$. Then

(i) if $y \in I, x < y$ and $f(y) \notin [a, y]$, then $[x, y]$ f-covers $[a, f(x)]$, and

(ii) if $y \in I, y < x$ and $f(y) \notin [a, y]$, then $[y, x]$ f-covers $[a, f(x)]$.

Proposition 2. *Let $f \in C^0(S^1, S^1)$. If $x \in s\alpha(y)$, then $f^n(x) \in s\alpha(y)$ for any positive integer n.*

Proof. Suppose that $x \in s\alpha(y)$. Then there exists a sequence of positive integers $\{n_i\}$ with $n_i \to \infty$ and a sequence of points $\{y_i\}$ such that $f^{n_1}(y_1) = y_0 = y, f^{n_i}(y_i) = y_{i-1}$, and $y_i \to x$. We have $f^{n_1-1}(f(y_1)) = f^{n_1}(y_1) = y_0 = y$, $f^{n_{i+1}}(f(y_{i+1})) = f(f^{n_{i+1}}(y_{i+1})) = f(y_i)$ and $f(y_i) \to f(x)$. Therefore $f(x) \in s\alpha(y)$. By induction, for any positive integer n, $f^n(x) \in s\alpha(y)$.

By definition of ω-limit point and Proposition 2, we have the following corollary.

Corollary 3. *If $x \in s\alpha(y)$, then $\omega(x) \subset s\alpha(y)$*

Proof. Suppose that $z \in \omega(x)$. Then there exists $n_i \to \infty$ such that $f^{n_i}(x) \to z$. By Proposition 2, we have $f^{n_i}(x) \in s\alpha(y)$. Since $s\alpha(y)$ is closed, we get $z \in s\alpha(y)$. Thus $\omega(x) \subset s\alpha(y)$.

Let $x \in S^1$ and $f \in C^0(S^1, S^1)$ be given. Then we will use the symbols $\alpha_+(x)$ (*resp.* $\alpha_-(x)$) to denote the set of all points $y \in S^1$ such that there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{x_i\}$ of points such that $x_i \to y, f^{n_i}(x_i) = x$ for every $i > 0$ and $y < \cdots < x_i < \cdots < x_2 < x_1$ (*resp.* $x_1 < x_2 < \cdots < x_i < \cdots < y$). It is clear that if $x \notin P$, then $\alpha(x) = \alpha_+(x) \cup \alpha_-(x)$.

Lemma 4.

(1) *if for some* $\epsilon > 0$, $x \in \alpha_-(y)$ *for every* $y \in (x-\epsilon, x)$, *then for every* $y \in (x-\epsilon, x)$, $x \in s\alpha(y)$.
(2) *if for some* $\epsilon > 0$, $x \in \alpha_+(y)$ *for every* $y \in (x, x+\epsilon)$, *then for every* $y \in (x, x+\epsilon)$, $x \in s\alpha(y)$.

Proof. Without loss of generality, we will prove part (1). Let $y \in (x-\epsilon, x)$. Since $x \in \alpha_-(y)$, we can find $y_1 \in (\frac{x-\epsilon}{2}, x)$ and $n_1 > 0$ with $f^{n_1}(y_1) = y$. By hypothesis, $x \in \alpha_-(y_1)$; thus we can find $y_2 \in (\frac{x-\epsilon}{2^2}, x)$ and $n_2 > 0$ with $f^{n_2}(y_2) = y_1$. Continuing in this way , we obtain a sequence $\{y_i\}$ and $n_i > 0$ with $y_i \in (\frac{x-\epsilon}{2^i}, x)$ and $f^{n_i}(y_i) = y_{i-1}$. We may assume that $n_i \to \infty$. Thus $x \in s\alpha(y)$.

Theorem A. *Let f be a continuous map of the circle S^1 to itself. Suppose that P is empty. Then $\overline{R} \subset S\Gamma$.*

Proof. Suppose that $x \in \overline{R} \setminus R$. Then there exists an arc (a, b) containing x such that $f^n(x) \notin (a, b)$ for any positive integer n. Since $x \in \overline{R}$, we may assume that there is a sequence $x_i \in R$ such that $a < x_1 < x_2 < \cdots < x_i < \cdots < x < b$ and $x_i \to x$. For each i, there exist n_i, m_i with $n_i < m_i$ such that either

$$x_{i-1} < f^{n_i}(x_i) < f^{m_i}(x_i) < x_i \tag{1}$$

or

$$x_i < f^{m_i}(x_i) < f^{n_i}(x_i) < x_{i+1}. \tag{2}$$

For $i = 1, 2, \cdots$ there exist sequences $y_i, z_i \in (x_{i-1}, x_{i+1})$ and n_i, m_i with $n_i < m_i$ such that

$$x_{i-1} < f^{n_i}(y_i) < y_i < x$$

and

$$x_{i-1} < z_i < f^{m_i}(z_i) < x_{i+1} < x$$

By Lemma 1,

$$[y_i, x] \quad f^{n_i}\text{-covers} \quad [a, f^{n_i}(y_i)]$$

and

$$[z_i, x] \quad f^{m_i}\text{-covers} \quad [f^{m_i}(z_i), b].$$

Consequently,

$$[x_{i-1}, x] \ \ f^{n_i}\text{-covers} \ \ [x_1, x_{i-1}] \quad \text{for each } i, \tag{$*$}$$

and

$$[x_{i-1}, x] \ \ f^{m_i}\text{-covers} \ \ [x_{i+1}, x] \quad \text{for each } i. \tag{$**$}$$

Now, let $K_i = [x_i, x]$ for all positive integer i, Then K_{i-1} f^{m_i}-covers K_{i+1}. Hence we may choose a closed arc L_1 in K_1 such that $f^{m_2}(L_1) = K_3$. Also, we can take a closed arc L_2 in L_1 such that $f^{m_2+m_4}(L_2) = K_5$. Continuing this process, we may take a closed arc $L_i \subset K_1$ such that $L_1 \supset L_2 \supset \cdots$ and $f^{\sum_{i=1}^{k} m_{2i}}(L_k) = K_{2k+1}$ for each $k = 1, 2, \cdots$. Let $y \in \bigcap_{i=1}^{\infty} L_i$. Then $x \in \omega(y)$.

Now, take N such that $x_{N-1} > y$. By $(*)$, for all $i \geq N$, there exists $y_i \in [x_{i-1}, x]$ such that $f^{n_i}(y_i) = y_{i-1}$ where $y_{N-1} = y$. Since $x_i \to x$, we have $y_i \to x$. We may assume that $n_i \to \infty$. Hence $x \in s\alpha(y)$. Thus $x \in \omega(y) \cap s\alpha(y) \subset S\Gamma$. It is easy to show that if $x \in R$, then $x \in S\Gamma$.

Let X be a compact metric space. A point $y \in X$ is said to be *almost periodic* if given an open set U_y containing y, one can find an integer $n > 0$ such that for any integer $q > 0$ there exists an integer r, $q \leq r \leq q + n$ with $f^r(y) \in U_y$. If for every $x \in \omega(y)$, we have that $\omega(x) = \omega(y)$, then $\omega(y)$ is said to be a *minimal set.* It is well known that for compact metric spaces a point y is almost periodic if and only if $y \in \omega(y)$ and $\omega(y)$ is a minimal set.

Lemma 5. *For any $f \in C^0(S^1, S^1)$, if $x \in AP$, then $x \in s\alpha(x)$*

Proof. Suppose that x is almost periodic. Then $x \in \omega(x)$ and $\omega(x)$ is a minimal set. Take a sequence $n_i \to \infty$, since $f^{n_i}(\omega(x)) = \omega(x)$, we can find a sequence $\{z_i\}$ with $z_i \to \infty$ such that $z_i \in \omega(x)$, $f^{n_i}(z_i) = z_{i-1}$ and $f^{n_1}(z_1) = x$. Let y be a limit point of this sequence. Then $y \in s\alpha(x)$ and $y \in \omega(x)$ since $\omega(x)$ is closed. In this case $\omega(x)$ is a minimal set. Hence $x \in \omega(y)$. By Corollary 3, $x \in s\alpha(x)$.

Let Y be an arc in S^1, and let $\overline{Y}$ denote the closure of Y as usual. A point $y \in S^1$ is called a *right-sided* (resp. *left-sided*) *accumulation point* of Y if for any $z \in S^1$, $(y, z) \cap Y \neq \phi$ (*resp.* $(z, y) \cap Y \neq \phi$).

The right-side closure $\overline{Y}_+$ (*resp.* left-side closure $\overline{Y}_-$) is the union of Y and the set of right-sided (*resp.* left-sided) accumulation points of Y. A point which is both a right-sided and a left-sided accumulation point of Y is called a *two-sided accumulation point* of Y.

Theorem B. *If $x \in \overline{P}_- \cap \overline{P}_+$, then $x \in s\alpha(x)$. If in addition x is not almost periodic, then for some $\epsilon > 0$, either $x \in s\alpha(y)$ for every $y \in (x-\epsilon, x]$ or $x \in s\alpha(y)$ for every $y \in [x, x+\epsilon)$.*

Proof. By Lemma 5, we may assume that x is not almost periodic. Then we can find an $\epsilon > 0$ such that if $n > 0$ is given, then for some $k > 0$, $f^{nk}(x) \notin (x-\epsilon, x+\epsilon)$. Since $x \in \overline{P}_- \cap \overline{P}_+$, we have sequences of periodic points $\{p_i\}$ and $\{q_i\}$ such that $\lim_{i\to\infty} p_i = x, \lim_{i\to\infty} q_i = x$. Without loss of generality, we may assume that $x - \epsilon < p_i < x < q_i < x + \epsilon$. Let n_i be the period of p_i and m_i the period of q_i. Let $l_i = n_i m_i$. Then we can find $k_i > 0$ such that $f^{l_i k_i}(x) \notin (x-\epsilon, x+\epsilon)$. Without loss of generality, we may assume that $f^{l_i k_i}(x) < x - \epsilon$ for every i. Since n_i is the period of p_i, $f^{l_i k_i}(p_i) = p_i$: thus

$$(p_i, x) f^{l_i k_i} - \text{covers } (x-\epsilon, p_i). \qquad (*)$$

Let $y \in (x-\epsilon, x)$. We will show that $x \in s\alpha(y)$. Since $\lim_{i\to\infty} p_i = x$, we can find a positive integer i_1 such that $x - \epsilon < y < p_{i_1} < x$. It follows from $(*)$ that there exists $z_{i_1} \in (p_{i_1}, x)$ with $f^{l_{i_1} k_{i_1}}(z_{i_1}) = y$. Since $z_{i_1} \in (p_{i_1}, x)$, we can find a positive integer i_2 such that $x - \epsilon < y < p_{i_1} < z_{i_1} < p_{i_2} < x$. It follows from $(*)$ that there exists $z_{i_2} \in (p_{i_2}, x)$ with $f^{l_{i_2} k_{i_2}}(z_{i_2}) = z_{i_1}$. Continuing in this way, it is possible to find i_j, p_{i_j} and z_{i_j} with

$$x - \epsilon < y < p_{i_1} < z_{i_1} < p_{i_2} < \cdots < p_{i_j} < z_{i_j} < \cdots < x$$

such that $f^{l_{i_j} k_{i_j}}(z_{i_j}) = z_{i_{j-1}}$. Thus $\lim_{j\to\infty} z_{i_j} = x$ since $\lim_{i\to\infty} p_i = x$. We may assume that $\lim_{j\to\infty} l_{i_j} k_{i_j} = \infty$. Thus $x \in s\alpha(y)$. To finish the proof of the theorem it suffices to show that we can find a point $v \in (x-\epsilon, x)$ and $n > 0$ with $f^n(v) = x$. The argument that we will give here is taken from [BY]. Let $g = f^{n_1}$ and $L = [p_1, x]$. Then since $g(p_1) = f^{n_1}(p_1) = p_1$, $K = g(L) \cup g^2(L) \cup \cdots$ is connected.

Let l_j denote the period of p_j with respect to g. Now for each $k = 1, 2, 3$, consider the sequence $\{g^{4l_j - k}(p_j)\}$ in K which has a sequence converging to some $u_k \in \overline{K}$. Then we know that $g^k(u_k) = x$. If $u_{k'} = u_{k''}$ for some $k' < k''$, then

$$g^{k''-k'}(x) = g^{k''-k'}(g^{k'}(u_{k'})) = g^{k''}(u_{k'}) = g^{k''}(u_{k''})) = x.$$

Since $x \notin P$, we must have that u_i are distinct points in $\overline{K}$ for each $i = 1, 2, 3$, so that, one of these points has to lie in K, say u_k. Then there are $v \in [p_1, x]$ and $t \geq 1$ such that $g^t(v) = u_k$. Therefore,

$$f^{(t+k)n_1}(v) = g^{k+t}(v) = g^k(g^t(v)) = g^k(u_k) = x.$$

The proof of theorem is complete.

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