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PRETOPOLOGICAL CONVERGENCE QUOTIENT MAPS

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1. Introduction

A convergence structure defined by Kent [4] is a correspondence between the filters on a given set X and the subsets of X which specifies which filters converge to points of X. This concept is defined to include types of convergence which are more general than that defined by specifying a topology on X. Thus, a convergence structure may be regarded as a generalization of a topology.

With a given convergence structure q on a set X, Kent [4] introduced associated convergence structures which are called a topological modification and a pretopological modification.

Also, Kent [6] introduced a convergence quotient map, which is a quotient map for a convergence space.

In this paper, we introduce notions of pretopological convergence quotient maps and topological convergence quotient maps, and investigate some properties on them.

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2. Preliminaries

A *convergence structure* q on a set X is defined to be a function from the set $F(X)$ of all filters on X into the set $P(X)$ of all subsets of X, satisfying the following conditions:

(1) $x \in q(\dot{x})$ for all $x \in X$;

(2) $\Phi \subset \Psi$ implies $q(\Phi) \subset q(\Psi)$;

(3) $x \in q(\Phi)$ implies $x \in q(\Phi \cap \dot{x})$,

where $\dot{x}$ denotes the principal ultrafilter containing $\{x\}$; Φ and Ψ are in $F(X)$. Then the pair (X, q) is called a *convergence space.* If $x \in q(\Phi)$, then we say that Φ *q-converges* to x. The filter $V_q(x)$ obtained by intersecting all filters which q-converge to x is called the *q-neighborhood filter* at x. If $V_q(x)$ q-converges to x for each $x \in X$, then q is said to be *pretopological* and the pair (X, q) is called a *pretopological convergence space.*

A convergence structure q is said to be *topological* if q is pretopological and for each $x \in X$, the filter $V_q(x)$ has a filter base $B_q(x)$ with the following property:

$$y \in G \in B_q(x) \text{ implies } G \in B_q(y).$$

In this case, the pair (X, q) is called a *topological convergence space.*

Let $C(X)$ be the set of all convergence structures on X, partially ordered as follows:

$$q_1 \leq q_2 \iff q_2(\Phi) \subset q_1(\Phi) \text{ for all } \Phi \in F(X).$$

If $q_1 \leq q_2$, then we say that q_1 is *coarser* than q_2, and q_2 is *finer* than q_1. By [5], we know that if q_1 is pretopological, then

$$q_1 \leq q_2 \iff V_{q_1}(x) \subset V_{q_2}(x) \text{ for all } x \in X.$$

Let (X, q) be a convergence space. Then

$$\tau(q) = \{U \subset X \mid U \in V_q(x) \text{ for all } x \in U\}$$

is said to be the *topology induced by a convergence structure q*.

While, let (X, τ) be a topological space and $N(x)$ the neighborhood system at $x \in X$ with respect to given topology τ. Then the *convergence structure* $c(\tau)$ *induced by* τ is defined as follows:

$$x \in c(\tau)(\Phi) \iff N(x) \subset \Phi.$$

for each $\Phi \in F(X)$. Then, $c(\tau)$ is a topological convergence structure on X.

For any $q \in C(X)$, we define the following related convergence structures, $\pi(q)$, and $\lambda(q)$:

(1) $x \in \pi(q)(\Phi)$ iff $V_q(x) \subset \Phi$.

(2) $x \in \lambda(q)(\Phi)$ iff $U_q(x) \subset \Phi$, where $U_q(x)$ is the filter generated by the sets $U \in V_q(x)$ which have the property: $y \in U$ implies $U \in V_q(y)$. In this case, $\pi(q)$ and $\lambda(q)$ are called the *pretopological modification* and the *topological modification* of q, and the pairs $(X, \pi(q))$ and $(X, \lambda(q))$ are called the *pretopological modification* and the *topological modification* of (X, q), respectively.

Let (X, q) be a convergence space and $N(x)$ the neighborhood system at $x \in X$ with respect to the topology $\tau(q)$.

Since $U_q(x) = V_{\lambda(q)}(x) = N(x)$, we know that

$$\lambda(q) = c(\tau(q)), \quad \tau(\lambda(q)) = \tau(q), \quad N(x) \subset V_q(x).$$

Proposition 1([4]). *(1) $\pi(q)$ is the finest pretopological convergence structure coarser than q.*

(2) $\lambda(q)$ is the finest topological convergence structure coarser than q.

(3) $\lambda(q) \leq \pi(q) \leq q$.

Let f be a map from X into Y and Φ a filter on X. Then $f(\Phi)$ means the filter generated by $\{f(F) \mid F \in \Phi\}$.([1])

Let f be a map from a convergence space (X, q) to a convergence space (Y, p). Then f is said to be *continuous* at a point $x \in X$, if the filter $f(\Phi)$ on Y p-converges to $f(x)$ for every filter Φ on X q-converging to x. If f is continuous at every point $x \in X$, then f is said to be continuous.

Let q and q' be in $C(X)$, and p and p' in $C(Y)$. Then, we know that if $q \leq q'$, $p \geq p'$ and $f\colon (X, q) \to (Y, p)$ is continuous, then $f\colon (X, q') \to (Y, p')$ is continuous.

Proposition 2 ([6]). *(1) If $f\colon (X, q) \to (Y, p)$ is continuous at $x \in X$, then $V_p(f(x)) \subset f(V_q(x))$.*

(2) If p is pretopological and $V_p(f(x)) \subset f(V_q(x))$, then $f\colon (X, q) \to (Y, p)$ is continuous at $x \in X$.

Proposition 3. *Let (X, q) and (Y, p) be convergence spaces. Then, $f\colon (X, \lambda(q)) \to (Y, \lambda(p))$ is continuous if and only if $f\colon (X, \tau(q)) \to (Y, \tau(p))$ is continuous.*

Proof. The proof is clear from $V_{\lambda(p)}(f(x)) = N(f(x))$ and $V_{\lambda(q)}(x) = N(x)$ for each $x \in X$, where $N(x)$ and $N(f(x))$ are the neighborhood systems at x and $f(x)$ with respect to $\tau(q)$ and $\tau(p)$, respectively.

Proposition 4. *If $f\colon (X, q) \to (Y, p)$ is continuous, then*

(1) $f\colon (X, \pi(q)) \to (Y, \pi(p))$ is continuous.

(2) $f\colon (X, \lambda(q)) \to (Y, \lambda(p))$ is continuous.

Proof. (1) Let $\Phi \in F(X)$ and $x \in \pi(q)(\Phi)$. Then $V_q(x) \subset \Phi$. Since $f\colon (X, q) \to (Y, p)$ is continuous at x, $V_p(f(x)) \subset f(V_q(x)) \subset f(\Phi)$. Thus, $f(x) \in \pi(p)(f(\Phi))$. This completes the proof.

(2) By Proposition 3, it is sufficient to show that $f:(X,\tau(q)) \to (Y,\tau(p))$ is continuous. Let $U \in \tau(p)$ and $x \in f^{-1}(U)$. Then $f(x) \in U$ and $U \in N(f(x)) \subset V_p(f(x))$. Since $f:(X,q) \to (Y,p)$ is continuous, $U \in f(V_q(x))$. Thus, $f^{-1}(U) \in V_q(x)$ and $f^{-1}(U) \in \tau(q)$. This completes the proof.

Let (X,q) be a convergence space, Y a nonempty set, and a map $f:(X,q) \to Y$ a surjection. The *convergence quotient structure* p on Y is the finest convergence structure on Y relative to which f is continuous. In this case, $f:(X,q) \to (Y,p)$ is called a *convergence quotient map* and the pair (Y,p) is called a *convergence quotient space.*

Proposition 5([6]). *If $f:(X,q) \to (Y,p)$ is a convergence quotient map, then, for each $y \in Y$, $V_p(y) = \cap\{f(V_q(x)) \mid x \in f^{-1}(y)\}$.*

3. Main Results

A surjection $f:(X,q) \to (Y,p)$ is called a *pretopological* (resp. *topological*) *convergence quotient map* if p is the finest pretopological (resp. topological) convergence structure on Y relative to which f is continuous.

Theorem 6. *Let $f:(X,q) \to (Y,p)$ be continuous. Then the following hold:*

(1) If q is pretopological and for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that $V_p(y) = f(V_q(x))$, then p is pretopological and $f:(X,q) \to (Y,p)$ is a convergence quotient map.

(2) If p is pretopological and $f:(X,q) \to (Y,p)$ is a convergence quotient map, then for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that $V_p(y) = f(V_q(x))$.

Proof. (1) Suppose that for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that $V_p(y) =$

$f(V_q(x))$. Since q is pretopological, we obtain $x \in q(V_q(x))$. From the continuity of $f:(X,q) \to (Y,p)$, we obtain $y = f(x) \in p(f(V_q(x))) = p(V_p(y))$ and so p is pretopological.

Let $f:(X,q) \to (Y,r)$ be a convergence quotient map. Then $p \leq r$. While, let $\Psi \in F(Y)$ and $y \in p(\Psi)$. Then $\Psi \supset V_p(y) = f(V_q(x))$ for some $x \in f^{-1}(y)$. Since $x \in q(V_q(x))$ and $f:(X,q) \to (Y,r)$ is a convergence quotient map, we obtain $y \in r(\Psi)$. Thus $p(\Psi) \subset r(\Psi)$ and so $p \geq r$. Finally, $p = r$. This completes the proof.

(2) Let $y \in Y$. Since p is pretopological, we obtain $y \in p(V_p(y))$. Since $f:(X,q) \to (Y,p)$ is a convergence quotient map, there exist $x \in f^{-1}(y)$ and $\Phi \in F(X)$ such that $V_p(y) \supset f(\Phi)$ and $x \in q(\Phi)$. Thus, $V_q(x) \subset \Phi$ and so $V_p(y) \supset f(V_q(x))$. Since $f:(X,q) \to (Y,p)$ is continuous, $V_p(y) \subset f(V_q(x))$. Finally, $V_p(y) = f(V_q(x))$. This completes the proof.

Theorem 7. *Let (Y,p) be pretopological and $f:(X,q) \to (Y,p)$ a surjection. Then the following are equivalent:*

(a) $f:(X,q) \to (Y,p)$ is a pretopological convergence quotient map.

(b) $\cap\{f(V_q(x)) \mid x \in f^{-1}(y)\} = V_p(y)$ for each $y \in Y$.

Proof. (a) $\Longrightarrow$ (b): It is clear that $f:(X,q) \to (Y,p)$ is continuous. Let $V(y) = \cap\{f(V_q(x)) \mid x \in f^{-1}(y)\}$ and define a convergence structure $r \in C(Y)$ as follows:

$$y \in r(\Psi) \iff V(y) \subset \Psi.$$

Since $V_r(y) = \cap\{\Psi \mid y \in r(\Psi)\} = \cap\{\Psi \mid V(y) \subset \Psi\} = V(y)$, we know that r is pretopological. Since $V_r(y) = V(y) \subset f(V_q(x))$ for all $x \in f^{-1}(y)$, we obtain that $f:(X,q) \to (Y,r)$ is continuous. Since $f:(X,q) \to (Y,p)$ is a pretopological convergence quotient map, we obtain $r \leq p$. While, since $f:(X,q) \to (Y,p)$ is continuous, $V_p(y) \subset f(V_q(x))$ for all $x \in f^{-1}(y)$ and so $V_p(y) \subset \cap\{f(V_q(x)) \mid x \in f^{-1}(y)\} = V(y) = V_r(y)$. Thus, $V_p(y) \subset V_r(y)$. Since p is pretopological, we obtain that $p \leq r$ and so $p = r$. This completes the proof.

(b) $\Longrightarrow$ (a): By the hypothesis, we know that $V_p(y) \subset f(V_q(x))$ for each $x \in f^{-1}(y)$. Since p is pretopological, we obtain that $f:(X,q) \to (Y,p)$ is continuous.

Let r be pretopological and $f:(X,q) \to (Y,r)$ continuous. Then $V_r(y) \subset f(V_q(x))$ for all $x \in f^{-1}(y)$. Thus, $V_r(y) \subset \cap\{f(V_q(x)) \mid x \in f^{-1}(y)\} = V_p(y)$. Since r is pretopological, we obtain $r \leq p$. This completes the proof.

Theorem 8. *If $f:(X,q) \to (Y,p)$ is a convergence quotient map, then the following hold:*

(1) $f:(X,q) \to (Y,\pi(p))$ and $f:(X,\pi(q)) \to (Y,\pi(p))$ are pretopological convergence quotient maps.

(2) $f:(X,q) \to (Y,\lambda(p))$ and $f:(X,\lambda(q)) \to (Y,\lambda(p))$ are topological convergence quotient maps.

Proof. (1) Since $\pi(p) \leq p$, we know that $f:(X,q) \to (Y,\pi(p))$ is continuous.

Let $r \in C(Y)$ be pretopological and $f:(X,q) \to (Y,r)$ continuous. We will show that $r \leq \pi(p)$. Let $y \in Y$. Then $V_r(y) \subset f(V_q(x))$ for all $x \in f^{-1}(y)$. By Proposition 5, $V_r(y) \subset \cap\{f(V_q(x)) \mid x \in f^{-1}(y)\} = V_p(y)$. Since r is pretopological, $r \leq p$. Consequently, $r \leq \pi(p)$. and so $f:(X,q) \to (Y,\pi(p))$ is a pretopological convergence quotient map.

Also, by Proposition 4, $f:(X,\pi(q)) \to (Y,\pi(p))$ is continuous. Thus, $f:(X,\pi(q)) \to (Y,\pi(p))$ is a pretopological convergence quotient map. (2) The proof is similar to (1).

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