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A SURVEY ON SYMPLECTIC GEOMETRY

JEONG KOO NAM

1. Introduction

A *symplectic manifold* is a pair (M, ω) consisting of a smooth manifold M and a non-degenerate closed 2-form ω on M. Locally, $\omega = \sum_{i,j=1}^{n} \omega_{ij} dx^i \wedge dx^j$ and $d\omega = 0$, where $n = \dim M$. The condition $d\omega = 0$ implies that locally $\omega = d\alpha$ with $\alpha = \sum_{k=1}^{n} \alpha_k dx^k$. There are three main sources of symplectic manifolds.

(A) Phase space for classical mechanical systems

$M = T^*(N)$ is the cotangent bundle of a smooth manifold N, i.e., the configuration space. $T(N)$ is the position-velocity space and $T^*(N)$ is the impulse-coordinate space. If $q^1, \cdots, q^n$ are local coordinate in N, $\partial_1, \cdots, \partial_n$ the corresponding tangent vector fields, $p_1, \cdots, p_n$ the corresponding coordinates in the fibres of $T^*(N)$ so that $< p, \partial_i >= p_i$, then the 1-form $\alpha = \sum_{i=1}^{n} p_i dq^i$ is a canonically defined form which is invariant under $\mathrm{Diff}(N)$ and $\omega = d\alpha$ is closed (even exact) and non-degenerate.

(B) Complex projective algebraic varieties

These are subvarieties of complex projective varieties $P_n(\mathbb{C})$ defined by algebraic equations with a canonical Kähler form obtained by restricting the canonical Kähler

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX form ω on $P_n(\mathbb{C})$. If $\omega = Re(w) + iIm(\omega)$, the real part $Re(\omega)$ of ω defines a Riemannian metric and the imaginary part $Im(\omega)$ defines a symplectic structure. These can be thought of as a *superization* of an even and an odd part. The even parts are Riemannian manifolds and the odd parts are symplectic manifolds.

(C) Coadjoint orbits of a Lie group

The study of coadjoint orbits of a Lie group is quite closely related to that of irreducible unitary representations of G. The relation between these was given by A. Kirillov [K1] in the case when G is a *nilpotent* Lie group. For a wide class of Lie groups including but not restricted to nilpotent groups, we get a similiar relation between the coadjoint orbits and the unitary dual $\hat{G}$(cf.[K2]). For a general theory of a Lie group, we refer to [He] and [V].

Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Then we have the coadjoint mapping

$$\mathrm{Ad}^*(x) : \mathfrak{g}^* \to \mathfrak{g}^*, \quad x \in G.$$

Here $\mathrm{Ad}^*(x)$ is the *contragredient* of the adjoint mapping $\mathrm{Ad}(x) = \mathfrak{g} \to \mathfrak{g}$. Therefore

$$(\mathrm{Ad}^*(x)l)(X) = l(\mathrm{Ad}(x^{-1})X), \quad l \in \mathfrak{g}^*,\ X \in \mathfrak{g}. \tag{1.1}$$

For the present time being, we fix a $\mathbb{R}$-linear form $l \in \mathfrak{g}^*$ on $\mathfrak{g}$ once and for all. We let

$$G_l := \{x \in G \mid \mathrm{Ad}^*(x)l = l\} \tag{1.2}$$

the stabilizer of the coadjoint action Ad^* of G on $\mathfrak{g}^*$ at l. Since G_l is a closed subgroup of G, G_l is a Lie subgroup of G. We denote by $\mathfrak{g}_l$ the Lie subalgebra of $\mathfrak{g}$

corresponding to G_l. For all $X \in \mathfrak{g}_l$, $Y \in \mathfrak{g}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} < Y, l > &= < Y, \mathrm{Ad}^*(\exp tX)l > \\ &= < \mathrm{Ad}(\exp(-tX))Y, l > \\ &= < e^{-t\,\mathrm{ad}\,X}(Y), l > \\ &= < X - t[X,Y] + \frac{t^2}{2}[X,[X,Y]] + \cdots, l > . \end{aligned}$$

Taking the derivative with respect to t at $t = 0$ yields

$$< [X,Y], l >= 0 \quad \text{for all } X \in \mathfrak{g}_l \text{ and } Y \in \mathfrak{g}. \tag{1.3}$$

In order to interpret this identity geometrically, we first define the skew-symmetric $\mathbb{R}$-bilinear form $B_l \in \Lambda^2(\mathfrak{g}^*)$ associated with l on $\mathfrak{g}$ by

$$B_l(X,Y) := < [X,Y], l >, \quad X, Y \in \mathfrak{g}. \tag{1.4}$$

Lemma 1.1. *Let* $\mathrm{rad}\, B_l$ *be the radical of* B_l *in* $\mathfrak{g}$, *i.e.,*

$$\mathrm{rad}\, B_l := \{X \in \mathfrak{g} \mid B_l(X,Y) = 0 \quad \textit{for all } Y \in \mathfrak{g}\}.$$

Then

$$\mathrm{rad}\, B_l = \mathfrak{g}_l = \{X \in \mathfrak{g} \mid \mathrm{ad}^*(X)l = 0\}.$$

Here ad^* *denotes the differential of* $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$.

Proof. By (1.3), $\mathfrak{g}_l \subseteq \mathrm{rad}\, B_l$. Conversely, suppose X is an element of $\mathfrak{g}$ such that $B_l(X,Y) = 0$ for all $Y \in \mathfrak{g}$. Then

$$\begin{aligned} < Y, l > &= < e^{-\mathrm{ad}(tX)}(Y), l > \qquad (*) \\ &= < \mathrm{Ad}(\exp(-tX))Y, l > \\ &= < Y, \mathrm{Ad}^*(\exp(tX))l > \end{aligned}$$

holds for all $Y \in \mathfrak{g}$. Therefore $\mathrm{Ad}^*(\exp(tX))l = l$. Thus $X \in \mathfrak{g}_l$. It remains to prove the identity $(*)$. Differentiating with respect to t at $t = 0$,

$$\frac{d}{dt}\bigg|_{t=0} < e^{-\mathrm{ad}(tX)}Y, l >= - < [X, Y], l >= 0.$$

Thus $< e^{-\mathrm{ad}(tX)}Y, l >$ is constant for t and taking $t = 0$, we get the identity $(*)$. For all $X, Y \in \mathfrak{g}$, we have the identities

$$\begin{aligned} < Y, \mathrm{ad}^*(X)l > &= \frac{d}{dt}\bigg|_{t=0} < Y, \mathrm{Ad}^*(\exp tX)l > \\ &= \frac{d}{dt}\bigg|_{t=0} < \mathrm{Ad}(\exp(-tX))Y, l > \\ &= - < \mathrm{ad}(X)Y, l > \\ &= - < [X, Y], l >= -B_l(X, Y). \end{aligned}$$

Therefore

$$\mathrm{rad}\, B_l = \{X \in \mathfrak{g} \mid \mathrm{ad}^*(X)l = 0\}. \qquad \square$$

Next we consider the smooth mapping $\Phi_l : G \to \mathfrak{g}^*$ defined by

$$\Phi_l(x) := \mathrm{Ad}^*(x)l, \quad x \in G. \tag{1.5}$$

The mapping Φ_l is called the *coadjoint orbit mapping defined by* $l \in \mathfrak{g}^*$. Clearly the coadjoint orbit $\mathrm{Ad}^*(G)l$ at l can be identified with the homogeneous manifold G/G_l. Let $\pi : G \to G/G_l$ be the canonical surjection. The bijective mapping

$$\dot{\Phi}_l : G/G_l \to \mathrm{Ad}^*(G)l, \quad \dot{\Phi}_l(xG_l) := \mathrm{Ad}^*(x)l, \quad x \in G \tag{1.6}$$

allows to identify G/G_l with $\mathrm{Ad}^*(G)l$ and to equip $\mathrm{Ad}^*(G)l$ with the structure of a C^∞-submanifold of $\mathfrak{g}^*$ in such a way that $\dot{\Phi}_l$ is a diffeomorphism. Since

$$\begin{aligned}\Phi_l(\exp tX) &= \mathrm{Ad}^*(\exp tX)l \\ &= e^{\mathrm{ad}^*(tX)}l \\ &= l + t\,\mathrm{ad}^*(X)l + \frac{t^2}{2}\,\mathrm{ad}^*(\mathrm{ad}^*(X))l + \cdots\end{aligned}$$

holds for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$,

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_l(\exp tX) = \mathrm{ad}^*(X)l.$$

Thus the differential $d\Phi_l(e) : \mathfrak{g} \to \mathfrak{g}^*$ of Φ_l at e is given by

$$d\Phi_l(e)(X) = \mathrm{ad}^*(X)l, \quad X \in \mathfrak{g}. \tag{1.7}$$

Consequently, we have, for all $X, Y \in \mathfrak{g}$,

$$\begin{aligned}B_l(Y,X) &=< [Y,X], l > \\ &= \; - < (\mathrm{ad}\,X)Y, l > \\ &= < Y, \mathrm{ad}^*(X)l > \\ &= < Y, d\Phi_l(e)(X) > .\end{aligned}$$

We observe that if $X \in \mathfrak{g}_l$, $< Y, d\Phi_l(e)(X) >= 0$ for all $Y \in \mathfrak{g}$ and hence $d\Phi_l(e)(X) = 0$. Therefore the image $d\Phi_l(e)(\mathfrak{g})$ is identified with the tangent space of the coadjoint orbit $\mathrm{Ad}^*(G)l$ at the point $l \in \mathfrak{g}^*$. It is easy to show that the tangent space of $\mathrm{Ad}^*(G)l$ at l is isomorphic to the quotient vector space $\mathfrak{g}/\operatorname{rad} B_l$ (over $\mathbb{R}$).

Let $\dot{B}_l$ denote the non-degenerate alternating $\mathbb{R}$-bilinear form on the quotient vector space $\mathfrak{g}/\operatorname{rad} B_l$ induced by B_l.

Lemma 1.2. *The tangent space of the coadjoint orbit* $\mathrm{Ad}^*(G)l$ *at* l *is a symplectic vector space with respect to* $\dot{B}_l$. *In particular, it has an even dimension over* $\mathbb{R}$.

Proof. It follows immediately from the previous argument. □

Now we are ready to prove that the coadjoint orbit $\Omega := \mathrm{Ad}^*(G)l$ is a symplectic manifold. We denote by $\tilde{X}$ the vector field on $\mathfrak{g}^*$ corresponding to $X \in \mathfrak{g}$. That means that we have a canonical map $\mathfrak{g} \to T_l(\Omega)$ sending $X \to \tilde{X}_l := \mathrm{ad}^*(X)l$. We observe that

$$\tilde{X}_l = \mathrm{ad}^*(X)l = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*(\exp tX)l.$$

According to Lemma 1.1, $\mathfrak{g}_l$ is precisely the kernel of this map. We may define a 2-form B_Ω on Ω by

$$B_\Omega(\tilde{X},\tilde{Y}) = B_\Omega(\mathrm{ad}^*(X)l, \mathrm{ad}^*(Y)l) := B_l(X,Y), \tag{1.8}$$

where $X, Y \in \mathfrak{g}$.

Theorem 1.3. B_Ω *is non-degenerate and closed. So the coadjoint orbit* $\Omega := \mathrm{Ad}^*(G)l$ *at* l *is a symplectic manifold.*

Proof. Let $\tilde{X}$ be a vector field on $\mathfrak{g}^*$ corresponding to $X \in \mathfrak{g}$ such that

$$B_\Omega(\tilde{X},\tilde{Y}) = 0 \quad \text{for all } \tilde{Y} \text{ with } X \in \mathfrak{g}.$$

Since $B_\Omega(\tilde{X},\tilde{Y}) =< [X,Y], l >= 0$ for all $Y \in \mathfrak{g}$, according to Lemma 1.1, $X \in \mathfrak{g}_l$. Thus $\tilde{X} = 0$. Hence B_Ω is non-degenerate. If $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ are three vector fields on $\Omega(X_1, X_2, X_3 \in \mathfrak{g})$, then

$$\begin{aligned}
&dB_\Omega(\tilde{X}_1,\tilde{X}_2,\tilde{X}_3)\\
&= \tilde{X}_1(B_\Omega(\tilde{X}_2,\tilde{X}_3)) - \tilde{X}_2(B_\Omega(\tilde{X}_1,\tilde{X}_3)) + \tilde{X}_3(B_\Omega(\tilde{X}_1,\tilde{X}_2))\\
&\quad - B_\Omega([\tilde{X}_1,\tilde{X}_2],\tilde{X}_3) + B_\Omega([\tilde{X}_1,\tilde{X}_3],\tilde{X}_2) - B_\Omega([\tilde{X}_2,\tilde{X}_3],\tilde{X}_1)\\
&= - < [[X_1,X_2],X_3] - [[X_1,X_3],X_2] + [[X_2,X_3],X_1], l >= 0
\end{aligned}$$

Therefore B_Ω is closed. Then (Ω, B_Ω) is a symplectic manifold. □

2. General Facts of Symplectic Geometry

On a symplectic manifold (M,ω) there is an isomorphism between vector and covector fields. Denoting the former space by $\mathrm{Vect}(M)$ and the latter by $A^1(M)$, we have

$$\mathrm{Vect}(M) \ni \xi \leftrightarrow \omega(\xi,\cdot) = \imath_\xi\omega \in A^1(M). \tag{2.1}$$

Let L denote the Lie derivative and let $\mathrm{Vect}(M,\omega)$ denote the set of Hamiltonian vector fields, that is,

$$\mathrm{Vect}(M,\omega) := \{\xi \in \mathrm{Vect}(M) \mid L_\xi\omega = 0\}. \tag{2.2}$$

Theorem 2.1. *A vector field ξ is Hamiltonian if and only if the corresponding covector field $\imath_\xi\omega$ is closed.*

Proof.

$$L_\xi\omega = \imath_\xi \circ d\omega + d\imath_\xi \circ \omega = 0 \Leftrightarrow d\imath_\xi\omega = 0.$$ □

If $\imath_\xi\omega$ is not only closed but exact, then we say that ξ is *strictly Hamiltonian*, and define its *Hamiltonian* f_ξ by :

$$\imath_\xi\omega = -df_\xi. \tag{2.3}$$

Conversely, if we start with a function f we obtain a Hamiltonian vector field ξ_f defined by : $\omega(\xi_f,\cdot) = -df$.

The space $C^\infty(M)$ is a Lie algebra with the *Poisson* bracket :

$$\{f,g\} = \omega(\xi_f,\xi_g) = -df(\xi_g) = -\xi_g f = \xi_f g.$$

Since

$$\xi_{\{f,g\}} = [\xi_f, \xi_g]$$

$f \to \xi_f : C^\infty(M) \to \mathrm{Vect}(M)$ is a Lie algebra homomorphism. Suppose a Lie group G acts on M and preserves ω. There is a homomorphism $\mathfrak{g} \to \mathrm{Vect}(M, \omega) : X \to \xi_X$. Suppose ξ_X is strictly Hamiltonian. If there is a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M) : X \to h_X$ so that the following diagram is commutative, the action is a *Poisson* action :

$$\begin{array}{ccc} & & \mathrm{Vect}(M,\omega) \\ \mathfrak{g} & & \uparrow \\ & & C^\infty(M). \end{array}$$

Theorem 2.2. *Suppose G and M are connected and G acts transitively on M and preserves ω. There exists :*

(1) *A covering $\tilde{M} \to M$*
(2) *A central extension $\tilde{G} \to G$*
(3) *A Poisson action of $\tilde{G}$ on $\tilde{M}$*

so that the diagram below is commutative.

$$\begin{array}{ccc} \tilde{G} \times \tilde{M} & \longrightarrow & \tilde{M} \\ \downarrow & & \downarrow \\ G \times M & \longrightarrow & M. \end{array}$$

Here the horizontal arrows denote actions and vertical arrows the natural projections. □

This leads to a characterization of homogeneous symplectic manifolds.

Theorem 2.3. *Each G-homogeneous mainfold M is locally isomorphic to a coadjoint orbit of G, or of a central extension $\tilde{G}$ of G.*

In view of the preceding Lemma one need only to consider Poisson actions. For these we introduce the *momentum* map $\mu : M \to \mathfrak{g}^*$ which is defined by :

$$< \mu(m), X > = h_X(m). \tag{2.4}$$

Theorem 2.3 follows from

Theorem 2.4. *If G is connected,*

$$\mu : M \to \mathfrak{g}^*$$

is covariant, that is, $\mu(gm) = \mathrm{Ad}^*(g)\mu(m)$.

Proof. Since G is connected it is enough to show infinitesimal covariance

$$-\xi_Y \mu(m) = \mathrm{Ad}^*(Y)\mu(M)$$

all $Y \in \mathfrak{g}$. Applying both sides to $X \in \mathfrak{g}$ we see that the desired equality becomes

$$-\xi_Y h_X = -h_{[Y,X]}$$

which is a consequence of the definition of the Poisson bracket. □

Momentum maps are also interesting in the infinite-dimensional situation. Here are some examples.

(1) $M = G = \mathbb{R}^{2n}$, $\tilde{M} = M$, $\tilde{G} = H_n$.

(2) $M = \mathbb{R}^{2n} \backslash \{0\}$, $G = Sp(2n, \mathbb{R})$. $\mathfrak{g}$ may be identified with the space of symmetric matrices $Sym(2n, \mathbb{R})$ which is isomorphic to $\mathfrak{g}^*$ in such a way that $< F, X >= tr(FX)$. In this case $\tilde{M} = M$, $\tilde{G} = G$, and $\mu(v) = vv^t$, v a column vector.

(3) $M = \mathbb{CP}^n$, $G = U(n+1)$. If $(z_0 : z_1 : \cdots : z_n)$ are homogeneous coordinates of z, then

$$\mu(z) = (h_{ij}), \quad h_{ij} = \frac{z_i \bar{z}_j}{\|z\|^2}.$$

(4) $M = T^*(N)$, $G = \mathrm{Diff}(N)$, $\mathfrak{g} = \mathrm{Vect}(N)$. Here μ is defined by :

$$< \mu(A), \xi > = < A, \xi(p(A)) >,$$

where $p : M \to N$ is the natural projection.

3. Polarization

Let $\mathfrak{g}$ be a Lie algebra over a field K. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. We recall that a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *subordinate* to $l \in \mathfrak{g}^*$ if $\mathfrak{h}$ forms a totally isotropic vector space of $\mathfrak{g}$ relative to the alternating K-bilinear form $B_l : \mathfrak{g} \times \mathfrak{g} \to K$ given by $B_l(X, Y) := < [X, Y], l >$ with $X, Y \in \mathfrak{g}$ associated with l on $\mathfrak{g}$, i.e.,

$$B_l|_{\mathfrak{h} \times \mathfrak{h}} = < [\mathfrak{h}, \mathfrak{h}], l > = 0.$$

Definition 3.1. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ subordinate to $l \in \mathfrak{g}^*$ is said to be a K-*polarization* of $\mathfrak{g}$ for l if $\mathfrak{h}$ is maximal among the totally isotropic vector subspaces of $\mathfrak{g}$ relative to B_l. In other words, if $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq P$ and $B_l|_{P \times P} = 0$, then we have $\mathfrak{h} = P$. In particular, each K-polarization $\mathfrak{h}$ of $\mathfrak{g}$ for $l \in \mathfrak{g}^*$ is subordinate to l and contains $\mathrm{rad}\, B_l = \mathfrak{g}_l$. Thus we have the inclusions,

$$\mathcal{Z} \hookrightarrow \mathrm{rad}\, B_l \hookrightarrow \mathfrak{h}, \tag{3.1}$$

where $\mathcal{Z}$ denotes the center of $\mathfrak{g}$.

Remark 3.2. *A maximal totally isotropic vector subspace of $\mathfrak{g}$ relative to B_l need not to be a K-polarization of $\mathfrak{g}$ for $l \in \mathfrak{g}^*$. If $\mathfrak{g}$ is finite dimensional over K, a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ subordinate to $l \in \mathfrak{g}^*$ of maximal dimension over K is not necessarily a K-polarization of $\mathfrak{g}$ for l. Moreover, it is not true that there exist K-polarizations of general Lie algebras $\mathfrak{g}$ over K for all K-linear forms $l \in \mathfrak{g}^*$. However, if $\mathfrak{g}$ is a nilpotent real Lie algebra, then there exist real polarizations of $\mathfrak{g}$ for arbitrary $\mathbb{R}$-linear forms l on $\mathfrak{g}$ and the subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ subordinate to $l \in \mathfrak{g}^*$ of maximal dimension over $\mathbb{R}$ are exactly the real polarizations of $\mathfrak{g}$ for l.*

Let E be a finite dimensional vector space over a field K and $B : E \times E \to K$ an alternating K-bilinear form on E. For any subset F of E, we let the vector subspace

$$F^{\perp} := \{x \in E \mid B(x,y) = 0 \text{ for all } y \in F\}$$

of E be the *orthogonal* subspace of E for F relative to B. In particular, $E^{\perp} = \operatorname{rad} B$.

Suppose that F is a vector subspace of E and define the K-linear mapping $f_F : E \to \left(F \big/ F \cap \operatorname{rad} B\right)^*$ by

$$f_F(x) := B(x, \cdot)|_F, \quad x \in E. \tag{3.2}$$

Then $\ker f_F = F^{\perp}$. Consequently we have

$$\dim_K F - \dim_K(F \cap \operatorname{rad} B) + \dim_K F^{\perp} = \dim_K E. \tag{3.3}$$

The vector subspace F of E is said to be *isotropic* or *coisotropic* relative to B if $F \subseteq F^{\perp}$ or $F^{\perp} \subseteq F$ respectively. In the *isotropic case* we have

$$2 \dim_K F \leqq \dim_K E + \dim_K(F \cap \operatorname{rad} B) \tag{3.4}$$

and in the *coisotropic case*

$$2 \dim_K F \geqq \dim_K E + \dim_K(F \cap \operatorname{rad} B). \tag{3.5}$$

Since $E^\perp = \operatorname{rad} B$ is an isotropic vector subspace of E, a *maximal* vector subspace F among the isotropic vector subspaces of E satisfies $\operatorname{rad} B \subseteq F$ and hence

$$2 \dim_K F \leqq \dim_K E + \dim_K \operatorname{rad} B. \tag{3.6}$$

Consequently the maximality property of F suggests that $\dim_K F$ actually attains the upper bound.

Lemma 3.2. *A vector subspace F of the finite dimensional vector space E over a field K is maximal among the isotropic vector subspaces of E relative to B if and only if*

$$2 \dim_K F = \dim_K E + \dim_K \operatorname{rad} B.$$

Proof. It will suffice to show that the dimension of each vector subspace F of E which is maximal among the isotropic subspaces of E relative to B attains the *upper bound*

$$\frac{1}{2}(\dim_K E + \dim_K \operatorname{rad} B).$$

Let $\dot{B}$ denote the non-degenerate alternating K-bilinear form induced by B on the quotient vector space $E/\operatorname{rad} B$ over K. Then $(E/\operatorname{rad} B, \dot{B})$ is a symplectic vector space over K. In particular, $\dim_K(E/\operatorname{rad} B)$ is an even positive integer. Choose a *Lagrangian* vector subspace L of $E/\operatorname{rad} B$. Since L is isotropic and coisotropic, it coincides with its orthogonal vector subspace of $E/\operatorname{rad} B$ for L relative to the symplectic form $\dot{B}$, i.e., $L = L^2$. It follows that

$$\dim_K L = \frac{1}{2} \dim_K(E/\operatorname{rad} B).$$

Denote by $\pi : E \to E/\operatorname{rad} B$ be the canonical surjection and let $F = \pi^{-1}(L)$ denote the preimage of L relative to π. Then F is an isotropic subspace of E relative to B

such that $\operatorname{rad} B \subseteq F$ satisfying the following

$$\begin{aligned}\dim_K F &= \frac{1}{2}\dim_K(E/\operatorname{rad} B) + \dim_K(\operatorname{rad} B)\\ &= \frac{1}{2}\dim_K E - \frac{1}{2}\dim_K(\operatorname{rad} B) + \dim_K(\operatorname{rad} B)\\ &= \frac{1}{2}\dim_K E + \frac{1}{2}\dim_K(\operatorname{rad} B).\end{aligned}$$

Since for each isotropic vector subspace P of E relative to B containing $\operatorname{rad} B$ the image $\pi(P)$ is isotropic in $E/\operatorname{rad} B$ relative to $\dot{B}$, the proof is complete. □

If we apply the preceding lemma to the case when E is a finite dimensional Lie algebra $\mathfrak{g}$ over a field K and $B \in \Lambda^2(E^*)$ is the alternating K-bilinear form $B_l : \mathfrak{g} \times \mathfrak{g} \to K$ defined by $B(X,Y) :=< [X,Y], l >$, then we get the following characterization of the K-polarization of $\mathfrak{g}$ for l.

Proposition 3.3. *Let $l \in \mathfrak{g}^*$ denote a K-linear form on the finite dimensional Lie algebra over a field K. For a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ subordinate to l the following conditions mutually equivalent :*

(1) *$\mathfrak{h}$ forms a K-polarization of $\mathfrak{g}$ for l.*
(2) *For any element $X \in \mathfrak{g}$ such that $B_l(X,Y) = 0$ holds for all $Y \in \mathfrak{h}$, we have $X \in \mathfrak{h}$.*
(3) *The orthogonal vector subspace $\mathfrak{h}^\perp$ of $\mathfrak{h}$ relative to B_l is contained in $\mathfrak{h}$.*
(4) $\dim_K \mathfrak{h} = \frac{1}{2}(\dim_K \mathfrak{g} + \dim_K(\operatorname{rad} B))$.

Remark 3.4. *For more details on symplectic manifolds, we refer to [A-B], [A-N] and [Au]. For instance, [A-B] discusses Floer homology, the moduli of pseudoholomorphic curves and the compactness of the moduli space, [A-N] deals with Legendre singularities and cobordisms, and [Au] deals with equivariant cohomology and toric manifolds.*

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DEPARTMENT OF MATHEMATICS, INHA UNIVERSITY, INCHEON 402-751, KOREA